

# On one multidimensional compressible nonlocal model of the dissipative QG equations

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**Abstract.** In this paper we study the Cauchy problem for one multidimensional compressible nonlocal model of the dissipative quasi-geostrophic equations. First, we obtain the local existence and uniqueness of the smooth non-negative solution or the strong solution in time. Secondly, for the sub-critical and critical case  $1 \leq \alpha \leq 2$ , we obtain the global existence and uniqueness results of the nonnegative smooth solution. Then, we prove the global existence of the weak solution for  $0 \leq \alpha \leq 2$  and  $\nu \geq 0$ . Finally, for the sub-critical case, we establish the global  $H^1$  and  $L^p, p > 2$ , decay rate of the smooth solution as  $t \rightarrow \infty$ .

**Keywords:** multidimensional compressible nonlocal model; dissipative quasi-geostrophic equations; super-critical case; sub-critical case

## §1 Introduction and main results

In this paper, we study the following Cauchy problem for one multidimensional compressible nonlocal model of the dissipative quasi-geostrophic equations:

$$\begin{cases} \partial_t \theta + \operatorname{div}(u\theta) + \nu(-\Delta)^{\frac{\alpha}{2}} \theta = 0, & x \in \mathbb{R}^N, t > 0, \\ u = \mathcal{R}\theta = (\mathcal{R}_1\theta, \mathcal{R}_2\theta, \dots, \mathcal{R}_N\theta), \\ \theta(x, 0) = \theta_0(x). \end{cases} \quad (1.1)$$

where  $\theta : \mathbb{R}^N \rightarrow \mathbb{R}$  is a scalar function of  $x$  and  $t$ , representing potential temperature,  $u(x, t)$  is the velocity field of fluid given by the Riesz transform  $u = \mathcal{R}\theta = (\mathcal{R}_1\theta, \mathcal{R}_2\theta, \dots, \mathcal{R}_N\theta)$ , defined by

$$\mathcal{R}_i(\theta)(x, t) = \frac{1}{(2\pi)^N} P.V. \int_{\mathbb{R}^N} \frac{(x_i - y_i)\theta(y, t)}{|x - y|^{N+1}} dy, i = 1, 2, \dots, N.$$

$\nu > 0$  is the dissipative coefficient and  $N \geq 1$ . We denote  $\Lambda^\alpha = (-\Delta)^{\frac{\alpha}{2}}$ , which is defined by the Fourier transform  $F(\Lambda^\alpha) = F((-\Delta)^{\frac{\alpha}{2}}) = |\xi|^\alpha$ . Moreover, for  $0 < \alpha \leq 2$ ,  $\Lambda^\alpha \theta$  is given (see

e.g. [14, 20]) by

$$\Lambda^\alpha \theta(x) = C_\alpha P.V. \int_{\mathbb{R}^N} \frac{\theta(x) - \theta(y)}{|x - y|^{N+\alpha}} dy, x \in \mathbb{R}^N \quad (1.2)$$

and, especially,  $\Lambda\theta$  is given by

$$\Lambda\theta(x) = C(N) P.V. \int_{\mathbb{R}^N} \frac{\theta(x) - \theta(y)}{|x - y|^{N+1}} dy = \operatorname{div} \mathcal{R}\theta(x), x \in \mathbb{R}^N. \quad (1.3)$$

One particular feature of system (1.1) is the relation with the dissipative quasi-geostrophic equations [12], which is easily derived by changing the incompressible velocity field  $u = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta)$  of surface QG equations into the compressible velocity field  $u = (\mathcal{R}_1\theta, \mathcal{R}_2\theta)$  for  $N = 2$ . The system (1.1) with  $N = 1$  and  $\nu \geq 0$  was displayed by Baker et al in [2] as a one-dimensional model of the 2D Vortex sheet problem, and was further investigated by D. Chae, et al [10] and Castro and Córdoba [8] and global existence, finite time singularities and ill-posedness was discussed therefore by using the theory of complex-value partial differential equations. However, the methods used by D. Chae, et al [10] and Castro and Córdoba [8] can not be applied to the present multidimensional problem. Hence, the main purpose of this paper is that we extend the results for the ND model (1.1) with  $N = 1$  given by D. Chae, et al [10] and Castro and Córdoba [8] to the general ND model (1.1) for  $N \geq 1$  by using completely different methods.

It should be pointed out that some multidimensional models related to the dissipative or inviscid quasi-geostrophic equations have been studied by many authors. P. Balodis and A. Córdoba [3] discuss the blow-up problem for a class of nonlinear and nonlocal transport equations by using an inequality for Riesz transforms. A. Castro, D. Cordoba et al [9] study heat transfer with a general fractional diffusion term of incompressible fluid in a porous medium governed by Darcy's law and obtain local and global wellposedness for the strong or weak solutions and the existence of the global attractor for the solutions. The global existence and finite time blow-up problems for the aggregation equations are studied in [21, 22, 23] for some different singular potentials. Recently, a porous medium equation with nonlocal diffusion effects given by an inverse fractional Laplacian operator, i.e,

$$\partial_t u = \nabla \cdot (u \nabla p), p = (-\Delta)^{-s} u, 0 < s < 1, \quad (1.4)$$

is studied by L. Caffarelli and L. Vazquez [7]. The existence of the global weak solution for the nonnegative and bounded initial data function with compact support or fast decay at infinity is proven. The existence and uniqueness of the local or global smooth solution remain open. Notice that the model (1.4) with  $s = \frac{1}{2}$  have a different sign from the ND model (1.1). We will show that the local or global existence of smooth solutions in time depends heavily upon the sign of the solutions or the sign of initial data. In particular, if the initial data  $u(t = 0) \leq 0$  is a smooth function, then the model (1.4) with  $s = \frac{1}{2}$  and initial data  $u(t = 0)$  has a smooth solution locally in time.

In this paper, we will investigate the general multidimensional compressible nonlocal flux (1.1) for the case  $0 \leq \alpha \leq 2$ ,  $\nu \geq 0$  and for the nonnegative initial data. Here the case  $\alpha = 1$  is called the critical case, the case  $1 < \alpha \leq 2$  is so-called sub-critical one and the case  $0 \leq \alpha < 1$  is super-critical one. Roughly speaking, the critical and super-critical cases are mathematically harder to deal with than the sub-critical case.

We now state our main results. First, we give the following local existence result for the smooth solution to the system (1.1) with  $\nu \geq 0$ .

**Theorem 1.1** *Let  $0 \leq \alpha \leq 2$  and  $\nu \geq 0$ . Assume that the initial data  $\theta_0 \geq 0$  and  $\theta_0 \in H^s(\mathbb{R}^N)$  for some positive integer  $s > \frac{N}{2} + 1$ . Then (1.1) has a unique smooth solution  $\theta \in C([0, T^*]; H^s(\mathbb{R}^N)) \cap C^1([0, T^*]; H^{s-2}(\mathbb{R}^N))$ , defined on  $[0, T^*)$ ,  $T^* = T(\|\theta_0\|_{H^s(\mathbb{R}^N)}) > 0$  is the maximal existence time. Moreover, if  $T^* < \infty$ , then*

$$\int_0^{T^*} \|\theta(\cdot, t)\|_{H^s} dt = \infty$$

or

$$\int_0^{T^*} (\|\Lambda\theta\|_{L^\infty} + \|\nabla\theta\|_{L^\infty} + \|\nabla u\|_{L^\infty}) dt = \infty.$$

**Remark 1.2** *The assumption that  $\theta_0 \geq 0$  in Theorem 1.1 plays a key role in obtaining local existence results on the smooth solution to the system (1.1). If we remove this assumption, in particular, we assume that the initial data  $\theta_0$  changes its sign in  $\mathbb{R}^N$ , we can not get the local existence of the smooth solution to the system (1.1) by using the method used in the proof of Theorem 1.1. This will be discussed in the future.*

If  $\nu > 0$  and  $0 < \alpha \leq 2$ , then we can obtain the local existence results on the strong solution to the system (1.1).

**Theorem 1.3** *Let  $0 < \alpha \leq 2$  and  $\nu > 0$ . Assume that  $\theta_0 \in L^p(\mathbb{R}^N)$  with  $p > 1$ . Then there exists a time  $T > 0$  such that the system (1.1) has a solution  $\theta$ , defined in  $[0, T]$ , satisfying  $\theta \in L^q([0, T]; L^p(\mathbb{R}^N))$  for any  $q > 1$ .*

*Further, assume that  $\theta_0 \in W^{l,p}(\mathbb{R}^N)$  with  $l > 1, p > 1$ . Then there exists a time  $T > 0$  such that the system (1.1) has a solution  $\theta$ , defined in  $[0, T]$ , satisfying  $\partial_x^\beta \theta \in L^q([0, T]; L^p(\mathbb{R}^N))$  for any  $q > 1$  and  $0 \leq |\beta| \leq l$ .*

Secondly, for the sub-critical case  $1 < \alpha \leq 2$  and  $\nu > 0$ , we have the following global existence and uniqueness results on strong or smooth solution to the system (1.1).

**Theorem 1.4** *Let  $1 < \alpha \leq 2$  and  $\nu > 0$  and suppose that  $\theta_0 \geq 0$ .*

- (i) *If  $\theta_0 \in H^2(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$  ( $p > \frac{N}{\alpha-1}$ ), then there exists a unique global solution  $\theta$  to (1.1) satisfying  $\theta \in C([0, \infty); H^2(\mathbb{R}^N))$ .*
- (ii) *If  $\theta_0 \in H^s(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$  ( $s > 0, p > \frac{N}{\alpha-1}$ ), then there exists a unique global solution  $\theta$  to (1.1) satisfying  $\theta \in C([0, \infty); H^s(\mathbb{R}^N))$ .*

For the critical case  $\alpha = 1$ , we have the following regularity result.

**Theorem 1.5** *Let  $\theta(x, t)$  be a solution to system (1.1). Then  $\theta$  verifies the level set energy inequalities, i.e., for every  $\lambda > 0$*

$$\int_{\mathbb{R}^N} \theta_\lambda^2(t_2, x) dx + 2 \int_{t_1}^{t_2} \int_{\mathbb{R}^N} |\Lambda^{\frac{1}{2}} \theta_\lambda|^2 dx dt \leq \int_{\mathbb{R}^N} \theta_\lambda^2(t_1, x) dx, \quad 0 < t_1 < t_2, \quad (1.5)$$

where  $\theta_\lambda = (\theta - \lambda)_+$ . It yields that for every  $t_0$  there exists  $\gamma > 0$  such that  $\theta$  is bounded in  $C^\gamma([t_0, \infty) \times \mathbb{R}^N)$ .

For the super-critical case  $0 < \alpha < 1$ , we have the following global existence results on weak solutions. A similar result holds for the case  $1 \leq \alpha \leq 2$ .

**Theorem 1.6** *Let  $T > 0$  be arbitrary. For every  $\theta_0 \in L^2(\mathbb{R}^N)$ ,  $\theta_0 \geq 0$  and  $0 < \alpha \leq 2$ , then there exists at least one weak solution of the system (1.1), satisfying*

$$\theta \in L^\infty([0, T]; L^2(\mathbb{R}^N)) \cap L^2([0, T]; H^{\frac{\alpha}{2}}(\mathbb{R}^N)). \quad (1.6)$$

Because the weak solution in Theorem 1.6 is not unique, we try to give a unique criterion on weak solution. We have the following regularity result for  $1 < \alpha \leq 2$ .

**Theorem 1.7** *Let  $T > 0$  be arbitrary. For every  $\theta_0 \in L^2(\mathbb{R}^N)$ ,  $\theta_0 \geq 0$  and  $1 < \alpha \leq 2$ , there exists a unique solution of (1.1) such that  $\theta \in L^\infty(0, T; L^2(\mathbb{R}^N)) \cap L^2(0, T; H^{\frac{\alpha}{2}}(\mathbb{R}^N)) \cap L^p(0, T; L^q(\mathbb{R}^N))$  for  $q > \frac{N}{\alpha-1}$  and  $\frac{1}{p} + \frac{N}{q\alpha} = 1 - \frac{1}{\alpha}$ .*

For the sub-critical case  $\alpha \in (1, 2]$ , we have the following decay rate for the global solution.

**Theorem 1.8** *Let  $\alpha \in (1, 2]$  and  $N > 2$ . Assume that  $\theta_0 \geq 0$ ,  $\theta_0 \in L^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$  and  $\Lambda\theta_0 \in L^2(\mathbb{R}^N)$ . Then the solution  $\theta(x, t)$  to the problem (1.1) have the following decay rate in time:*

$$(i) \quad \|\theta(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{2}(\frac{N+2-2\alpha}{\alpha}-\epsilon)}; \quad (1.7)$$

$$(ii) \quad \|\theta(t)\|_{L^p} \leq C(1+t)^{-\frac{N(p-2)}{2p\alpha}}, p > 2; \quad (1.8)$$

$$(iii) \quad \|\nabla\theta(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{2}(\frac{N+2-2\alpha}{\alpha}-\epsilon)}. \quad (1.9)$$

Here  $C$  is a positive constant and  $\epsilon$  is sufficiently small positive constant.

We also mention that, for the incompressible quasi-geostrophic equations and the related models, there are a lot of results on the existence, uniqueness and the regularity (see [18, 11, 16, 1, 29, 30, 31, 32] and therein references). Nonlinear evolution problems involving the fractal Laplacian describing the anomalous diffusion, called the  $\alpha$ -stable Lévy diffusion, have been extensively studied in the mathematical and physical literature (see [19, 11, 22, 16, 6] and therein references).

Before ending this section, we give some preliminary Lemmas and recall some properties of the fractional operator  $\Lambda^\alpha$ , which will be used later.

First, we need the following basic calculus inequality (see [17, 28]).

**Lemma 1.9** *For  $s \geq 1$  and  $1 < r < p \leq \infty$ ,*

$$\|\Lambda^s(uv)\|_{L^r} \leq C(\|u\|_{L^p}\|\Lambda^s v\|_{L^q} + \|v\|_{L^p}\|\Lambda^s u\|_{L^q}), \quad (1.10)$$

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^2} \leq C(\|\nabla f\|_{L^\infty}\|\Lambda^{s-1}g\|_{L^2} + \|g\|_{L^\infty}\|\Lambda^s f\|_{L^2}) \quad (1.11)$$

where  $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$  and  $C$  is a constant.

We also need the following inequality for the Riesz potential (see [27]).

**Lemma 1.10** *Assume  $1 < q < p < +\infty$ ,  $0 < \delta < N$  and  $\frac{1}{q} = \frac{1}{p} + \frac{\delta}{N}$ . Then there exists a constant  $C > 0$  such that*

$$\|\Lambda^{-\delta} f\|_{L^p} \leq C \|f\|_{L^q}. \quad (1.12)$$

Secondly, we recall the following point-wise estimate and positive Lemma (see [14, 20, 9]).

**Lemma 1.11** *Let  $s \in [0, 2]$ ,  $\beta \geq -1$  and  $\theta \in \mathcal{S}(\Omega)$ , when  $\Omega = \mathbb{R}^N$ . Then the following point-wise inequality holds:*

$$|\theta(x)|^\beta \theta(x) \Lambda^s \theta(x) \geq \frac{1}{\beta + 2} \Lambda^s |\theta(x)|^{\beta+2}. \quad (1.13)$$

**Lemma 1.12** *Suppose that  $s \in [0, 2]$ ,  $x \in \mathbb{R}^N$  and  $\theta, \Lambda^s \theta \in L^p$ , with  $p \in (1, +\infty)$ . Then*

$$\int_{\mathbb{R}^N} |\theta|^{p-2} \theta \Lambda^s \theta dx \geq \frac{2}{p} \int_{\mathbb{R}^N} (\Lambda^{\frac{s}{2}} |\theta|^{\frac{p}{2}})^2 dx \geq 0. \quad (1.14)$$

Next, we recall the basic properties of the fractional operator  $\Lambda^\alpha$  (see [27]) and the Riesz transform.

**Lemma 1.13**

- (i)  $\Lambda \nabla = \nabla \Lambda$ ;
- (ii)  $\Lambda^\alpha \Lambda^\beta = \Lambda^{\alpha+\beta}$ ;
- (iii)  $C^{-1} \|\nabla f\|_{L^2} \leq \|\Lambda f\|_{L^2} \leq C \|\nabla f\|_{L^2}$ ;
- (iv)  $\|\mathcal{R}f\|_{L^p} \leq C \|f\|_{L^p}$ ,  $1 < p < \infty$

for some positive constant  $C$ .

Finally, we also give another property of the Riesz transform and its proof.

**Proposition 1.14** *Let  $\phi$  be a continuous function on  $\mathbb{R}^N$ . For any  $f \in \mathcal{S}(\mathbb{R}^N)$  ( $\mathcal{S}(\mathbb{R}^N)$  is the Schwartz class on  $\mathbb{R}^N$ ), we have*

$$\int_{\mathbb{R}^N} \phi(x) f(x) \mathcal{R}f(x) dx = \frac{C_N}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(x-y)[\phi(x) - \phi(y)]}{|x-y|^{N+1}} f(x) f(y) dx dy. \quad (1.15)$$

where  $C_N = (2\pi)^{-N}$ .

**Proof .** Denote  $\tilde{f}_\epsilon(x) = C_N \int_{\mathbb{R}^N, |x-y| \geq \epsilon} \frac{(x-y)f(y)}{|x-y|^{N+1}} dy$ ,  $F_\epsilon(x) = \phi(x)f(x)\tilde{f}_\epsilon(x)$  and  $\bar{f}(x) = \sup_{\epsilon \geq 0} |\tilde{f}_\epsilon(x)|$ . It follows from the singular integral theory of Calderon-Zygmund [15] that

$$\tilde{f}_\epsilon(x) \rightarrow \mathcal{R}f(x), \quad \text{for a.e. } x \in \mathbb{R}^N$$

and

$$\|\tilde{f}\|_{L^p(\mathbb{R}^N)} \leq C_p \|f\|_{L^p(\mathbb{R}^N)}.$$

Therefore, we have  $F_\epsilon(x) \rightarrow \phi(x)f(x)\mathcal{R}f(x)$ , for a.e.  $x \in \mathbb{R}^N$  and  $|F_\epsilon(x)| \leq G(x)$ , where  $G(x) = |\phi(x)f(x)|\bar{f}(x)$  satisfies

$$\begin{aligned} \|G(x)\|_{L^1(\mathbb{R}^N)} &\leq \|\bar{f}(x)\|_{L^p(\mathbb{R}^N)} \|\phi(x)f(x)\|_{L^q(\mathbb{R}^N)} \\ &\leq C_p \|f(x)\|_{L^p(\mathbb{R}^N)} \|\phi(x)f(x)\|_{L^q(\mathbb{R}^N)} < +\infty. \end{aligned}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $p > 1$ .

Using the Lebesgue Dominated Convergence Theorem, we have

$$\begin{aligned} \int_{\mathbb{R}^N} \phi(x) f(x) \mathcal{R}(f) dx &= \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} f(x) \phi(x) \tilde{f}_\epsilon(x) dx \\ &= C_N \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} f(x) \phi(x) \int_{\mathbb{R}^N, |x-y| \geq \epsilon} \frac{(x-y)f(y)}{|x-y|^{N+1}} dy dx. \end{aligned} \quad (1.16)$$

Note that

$$\begin{aligned} & \int_{\mathbb{R}^N} |f(y)| \left( \int_{\mathbb{R}^N, |x-y| \geq \epsilon} \frac{|(x-y)f(x)\phi(x)|}{|x-y|^{N+1}} dx \right) dy \\ & \leq \int_{\mathbb{R}^N} |f(y)| \left( \int_{\mathbb{R}^N} \frac{2|f(x)\phi(x)|}{\epsilon + |x-y|^N} dx \right) dy \\ & \leq 2 \|\phi(x)f(x)\|_{L^q} \|(\epsilon + |x|^N)^{-1}\|_{L^p} \int_{\mathbb{R}^N} |f(y)| dy \\ & = C \|f(y)\|_{L^1(\mathbb{R}^N)} \|\phi(x)f(x)\|_{L^q(\mathbb{R}^N)} < \infty, \end{aligned}$$

for each fixed  $\epsilon > 0$  since  $f \in L^1(\mathbb{R}^N)$ ,  $\phi f \in L^q(\mathbb{R}^N)$  by our assumption, and  $C \equiv 2\|(\epsilon + |x|^N)^{-1}\|_{L^p(\mathbb{R}^N)} < \infty$  for  $p > 1$  ( $\frac{1}{p} + \frac{1}{q} = 1$ ). Thus Fubini's Theorem implies that

$$\begin{aligned} & C_N \int_{\mathbb{R}^N} f(x) \phi(x) \int_{\mathbb{R}^N, |x-y| \geq \epsilon} \frac{(x-y)f(y)}{|x-y|^{N+1}} dy dx \\ & = C_N \int_{\mathbb{R}^N} \int_{\mathbb{R}^N, |x-y| \geq \epsilon} f(x) \phi(x) \frac{(x-y)f(y)}{|x-y|^{N+1}} dy dx, \end{aligned} \quad (1.17)$$

for each fixed  $\epsilon > 0$ . Furthermore, by renaming the variables in the integration, we can rewrite 1/2 of the integral on the right hand side of (1.17) as follows:

$$\begin{aligned} & \frac{C_N}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N, |x-y| \geq \epsilon} f(x) f(y) \frac{(x-y)\phi(x)}{|x-y|^{N+1}} dy dx \\ & = -\frac{C_N}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N, |x-y| \geq \epsilon} f(x) f(y) \frac{(x-y)\phi(y)}{|x-y|^{N+1}} dx dy, \end{aligned}$$

which implies that

$$\begin{aligned} & C_N \int_{\mathbb{R}^N} \int_{\mathbb{R}^N, |x-y| \geq \epsilon} f(x) f(y) \frac{(x-y)\phi(x)}{|x-y|^{N+1}} dy dx \\ & = \frac{C_N}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N, |x-y| \geq \epsilon} f(x) f(y) \frac{(x-y)[\phi(x) - \phi(y)]}{|x-y|^{N+1}} dx dy. \end{aligned} \quad (1.18)$$

Since  $f \in \mathcal{S}(\mathbb{R}^N)$  and  $\phi(x)$  is continuous on  $\mathbb{R}^N$ , it is obvious that

$$f(x) f(y) \frac{(x-y)[\phi(x) - \phi(y)]}{|x-y|^{N+1}} \in L^1(\mathbb{R}^{2N}).$$

Using the Lebesgue Dominated Convergence Theorem, we have

$$\begin{aligned} & \frac{C_N}{2} \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N, |x-y| \geq \epsilon} f(x) f(y) \frac{(x-y)[\phi(x) - \phi(y)]}{|x-y|^{N+1}} dx dy \\ & = \frac{C_N}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(x) f(y) \frac{(x-y)[\phi(x) - \phi(y)]}{|x-y|^{N+1}} dx dy. \end{aligned} \quad (1.19)$$

Proposition 1.14 now follows from (1.16)-(1.19).  $\square$

The rest of this article is organized as follows. In section 2, we prove the local existence and uniqueness of smooth non-negative solutions or the strong solution to the system (1.1) with or

without the dissipation term. Section 3 is devoted to the global existence and uniqueness of smooth or strong solutions for the sub-critical and critical cases. In section 4, we prove the global existence of weak solution and give one Leray-Prodi-Serrin condition on uniqueness of the strong solution for the sub-critical case. An example that the non-positive solution to the system (1.1) with  $\nu = 0$  can not be global in time is also given. Finally, we establish the decay rate of the smooth solution to the system (1.1) in the sub-critical case as  $t \rightarrow \infty$  in section 5.

## §2 Local existence: proofs of Theorems 1.1 and 1.3

In this section, we give proofs of Theorems 1.1 and 1.3.

**The proof of Theorem 1.1:** For  $\alpha = 2$  and  $\nu > 0$ , the existence and uniqueness of local smooth solution is standard. We will prove our results for the case  $0 \leq \alpha < 2$  and  $\nu \geq 0$  by using the regularization method. We consider the regularization system as follows:

$$\partial_t \theta^\epsilon + u^\epsilon \cdot \nabla \theta^\epsilon + \theta^\epsilon \operatorname{div} u^\epsilon = -\nu \Lambda^\alpha \theta^\epsilon + \epsilon \Delta \theta^\epsilon, x \in \mathbb{R}^N, t > 0, \quad (2.1)$$

$$u^\epsilon = \mathcal{R} \theta^\epsilon, \operatorname{div} u^\epsilon = \operatorname{div} \mathcal{R} \theta^\epsilon = \Lambda \theta^\epsilon, x \in \mathbb{R}^N, t > 0, \quad (2.2)$$

$$\theta^\epsilon(x, t) = \theta_0(x), x \in \mathbb{R}^N, \quad (2.3)$$

which, using the semigroup theory, can be re-written into the equivalent integral form:

$$\theta^\epsilon(x, t) = e^{\epsilon t \Delta} \theta_0(x) + \int_0^t e^{\epsilon(t-\tau) \Delta} (-u^\epsilon \cdot \nabla \theta^\epsilon - \theta^\epsilon \operatorname{div} u^\epsilon - \nu \Lambda^\alpha \theta^\epsilon)(x, \tau) d\tau, x \in \mathbb{R}^N, t > 0. \quad (2.4)$$

Notice that the singular integral operator  $\Lambda^\alpha, 0 \leq \alpha < 2$ , is of the order  $\alpha, 0 \leq \alpha < 2$ , and, hence, the system (2.1)-(2.3) is a parabolic one of the order 2 with nonlocal singular integrals, which are an operators from  $H^s$  to  $H^s$  for any  $s \geq 0$ . Thus, it is easy to prove, by the standard parabolic theory and using the fact that  $\|u^\epsilon\|_{H^s} \leq C \|\theta^\epsilon\|_{H^s}, \|\Lambda \theta^\epsilon\|_{H^s} \leq C \|\theta^\epsilon\|_{H^{s+1}}$ , that, for any  $\epsilon > 0$ , there exists  $T^\epsilon > 0$  such that the system (2.1)-(2.3) has a unique smooth solution  $\theta \in C(0, T^\epsilon; H^s(\mathbb{R}^N)) \cap C^1(0, T^\epsilon; H^{s-2}(\mathbb{R}^N))$ . Moreover, using the fact that, if  $\theta(x_0, t_0) = \min_{x \in \mathbb{R}^N, t \geq 0} \theta^\epsilon(x, t)$ , then

$$\Lambda^\alpha \theta^\epsilon(x_0, t_0) = C_\alpha P.V. \int_{\mathbb{R}^N} \frac{\theta^\epsilon(x_0, t_0) - \theta^\epsilon(y, t)}{|x_0 - y|^{N+\alpha}} dy \leq 0,$$

it is easy to prove that, if  $\theta_0(x) \geq 0$  in  $\mathbb{R}^N$ , then  $\theta^\epsilon \geq 0$  in  $\mathbb{R}^N \times [0, T^\epsilon]$ .

In the following we want to prove that, if  $\theta_0 \in H^s(\mathbb{R}^N), s > \frac{N}{2}$ , satisfying  $\theta_0 \geq 0$ , then there exist a time  $T_0 = T_0(\theta_0) > 0$  and a positive constant  $M$ , independent of  $\epsilon$  such that, for all  $\epsilon > 0$ , the solution  $\theta^\epsilon$  of the system (2.1)-(2.3) satisfies  $\theta^\epsilon \geq 0$  and

$$\sup_{0 \leq t \leq T_0} \|\theta^\epsilon(x, t)\|_{H^s(\mathbb{R}^N)} + \sup_{0 \leq t \leq T_0} \|\partial_t \theta^\epsilon(x, t)\|_{H^{s-2}(\mathbb{R}^N)} + \int_0^{T_0} \|\theta^\epsilon(x, t)\|_{H^{s+\frac{\alpha}{2}}(\mathbb{R}^N)} dt \leq M. \quad (2.5)$$

Multiplying the equation (2.1) by  $\Lambda^{2s}\theta^\epsilon$  and integrating by parts, we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\Lambda^s \theta^\epsilon\|_{L^2}^2 + \nu \|\Lambda^{s+\frac{\alpha}{2}} \theta^\epsilon\|_{L^2}^2 + \epsilon \|\Lambda^{s+1} \theta^\epsilon\|_{L^2}^2 \\
&= - \int_{\mathbb{R}^N} \Lambda^s (u^\epsilon \cdot \nabla \theta^\epsilon + \theta^\epsilon \operatorname{div} u^\epsilon) \cdot \Lambda^s \theta^\epsilon dx \\
&= - \int_{\mathbb{R}^N} \Lambda^s (u^\epsilon \cdot \nabla \theta^\epsilon) \Lambda^s \theta^\epsilon dx - \int_{\mathbb{R}^N} \Lambda^s (\theta^\epsilon \Lambda \theta^\epsilon) \Lambda^s \theta^\epsilon dx \\
&\equiv I_1 + I_2.
\end{aligned} \tag{2.6}$$

For the first term  $I_1$ , we have

$$\begin{aligned}
I_1 &= - \int_{\mathbb{R}^N} \Lambda^s (u^\epsilon \cdot \nabla \theta^\epsilon) \Lambda^s \theta^\epsilon dx \\
&= - \int_{\mathbb{R}^N} u^\epsilon \cdot \nabla \Lambda^s \theta^\epsilon \Lambda^s \theta^\epsilon dx - \int_{\mathbb{R}^N} [\Lambda^s (u^\epsilon \cdot \nabla \theta^\epsilon) - u^\epsilon \cdot \Lambda^s (\nabla \theta^\epsilon)] \Lambda^s \theta^\epsilon dx \\
&= - \int_{\mathbb{R}^N} u^\epsilon \cdot \nabla \frac{|\Lambda^s \theta^\epsilon|^2}{2} dx - \int_{\mathbb{R}^N} [\Lambda^s (u^\epsilon \cdot \nabla \theta^\epsilon) - u^\epsilon \cdot \Lambda^s (\nabla \theta^\epsilon)] \Lambda^s \theta^\epsilon dx \\
&= \int_{\mathbb{R}^N} \Lambda \theta^\epsilon \frac{|\Lambda^s \theta^\epsilon|^2}{2} dx - \int_{\mathbb{R}^N} [\Lambda^s (u^\epsilon \cdot \nabla \theta^\epsilon) - u^\epsilon \cdot \Lambda^s (\nabla \theta^\epsilon)] \Lambda^s \theta^\epsilon dx \\
&\leq \frac{1}{2} \|\Lambda \theta^\epsilon\|_{L^\infty} \int_{\mathbb{R}^N} |\Lambda^s \theta^\epsilon|^2 dx + \|\Lambda^s (u^\epsilon \cdot \nabla \theta^\epsilon) - u^\epsilon \cdot \Lambda^s (\nabla \theta^\epsilon)\|_{L^2} \|\Lambda^s \theta^\epsilon\|_{L^2} \\
&\leq \frac{1}{2} \|\Lambda \theta^\epsilon\|_{L^\infty} \int_{\mathbb{R}^N} |\Lambda^s \theta^\epsilon|^2 dx + C(\|\nabla u^\epsilon\|_{L^\infty} \|\Lambda^{s-1} \nabla \theta^\epsilon\|_{L^2} + \|\nabla \theta^\epsilon\|_{L^\infty} \|\Lambda^s u^\epsilon\|_{L^2}) \|\Lambda^s \theta^\epsilon\|_{L^2} \\
&\leq C \|\Lambda \theta^\epsilon\|_{H^{s-1}} \int_{\mathbb{R}^N} |\Lambda^s \theta^\epsilon|^2 dx + C(\|\nabla u^\epsilon\|_{H^{s-1}} \|\Lambda^{s-1} \nabla \theta^\epsilon\|_{L^2} + \|\nabla \theta^\epsilon\|_{H^{s-1}} \|\Lambda^s u^\epsilon\|_{L^2}) \|\Lambda^s \theta^\epsilon\|_{L^2} \\
&\leq C \|\Lambda^s \theta^\epsilon\|_{L^2} \int_{\mathbb{R}^N} |\Lambda^s \theta^\epsilon|^2 dx + C \|\Lambda^s \theta^\epsilon\|_{L^2} \|\Lambda^s \theta^\epsilon\|_{L^2} \|\Lambda^s \theta^\epsilon\|_{L^2} \leq C \|\Lambda^s \theta^\epsilon\|_{L^2}^3.
\end{aligned} \tag{2.7}$$

Here  $C$  is a positive constant independent of  $\epsilon$ , and we have used the fact that  $\|\nabla f\|_{H^s} \leq C \|\Lambda f\|_{H^s}$ ,  $\|u^\epsilon\|_{H^s} \leq C \|\theta^\epsilon\|_{H^s}$  for  $u^\epsilon = \mathcal{R} \theta^\epsilon$  and the inequality (1.11).

For the second term  $I_2$ , using the fact that  $\theta^\epsilon \geq 0$ , the pointwise estimate (1.13) with  $\beta = 0$  for the operator  $\Lambda$  and the inequality (1.11), we have

$$\begin{aligned}
I_2 &= - \int_{\mathbb{R}^N} \theta^\epsilon \Lambda (\Lambda^s \theta^\epsilon) \Lambda^s \theta^\epsilon dx - \int_{\mathbb{R}^N} [\Lambda^s (\theta^\epsilon \Lambda \theta^\epsilon) - \theta^\epsilon \Lambda^s (\Lambda \theta^\epsilon)] \Lambda^s \theta^\epsilon dx \\
&\leq - \int_{\mathbb{R}^N} \theta^\epsilon \Lambda \frac{|\Lambda^s \theta^\epsilon|^2}{2} dx + C(\|\Lambda \theta^\epsilon\|_{L^\infty} + \|\nabla \theta^\epsilon\|_{L^\infty}) \|\Lambda^s \theta^\epsilon\|_{L^2}^2 \\
&= \frac{1}{2} \int_{\mathbb{R}^N} \Lambda \theta^\epsilon |\Lambda^s \theta^\epsilon|^2 dx + C(\|\Lambda \theta^\epsilon\|_{L^\infty} + \|\nabla \theta^\epsilon\|_{L^\infty}) \|\Lambda^s \theta^\epsilon\|_{L^2}^2 \\
&\leq C(\|\Lambda \theta^\epsilon\|_{L^\infty} + \|\nabla \theta^\epsilon\|_{L^\infty}) \|\Lambda^s \theta^\epsilon\|_{L^2}^2 \\
&\leq C \|\Lambda^s \theta^\epsilon\|_{L^2}^3.
\end{aligned} \tag{2.8}$$

Combining (2.6) with (2.7) and (2.8), one have

$$\frac{1}{2} \frac{d}{dt} \|\Lambda^s \theta^\epsilon\|_{L^2}^2 + \nu \|\Lambda^{s+\frac{\alpha}{2}} \theta^\epsilon\|_{L^2}^2 + \epsilon \|\Lambda^{s+1} \theta^\epsilon\|_{L^2}^2 \leq C \|\Lambda^s \theta^\epsilon\|_{L^2}^3, \tag{2.9}$$

which claims that there exist a time  $T_0 > 0$  and a constant  $M(T) > 0$ , independent of  $\epsilon$ , such that  $\sup_{0 \leq t \leq T_0} \|\Lambda^s \theta^\epsilon(\cdot, t)\|_{L^2} + \int_0^{T_0} \|\Lambda^{s+\frac{\alpha}{2}} \theta^\epsilon(\cdot, t)\|_{L^2}^2 dt \leq M(T_0)$ . Then, by the equations (2.1) and (2.2), the uniform estimate for  $\partial_t \theta^\epsilon$  with respect to  $\epsilon$  can be obtained easily.



Now combining the above estimates with the compactness argument, letting  $\epsilon \rightarrow 0$ , we obtain the desired results on the local smooth solutions to the system (1.1). Moreover, it follows from (2.9) that, if  $0 < T < \infty$ ,  $T$  is the maximal existence time of the solution to the system (1.1), then  $\int_0^T \|\theta(\cdot, t)\|_{H^s} dt = \infty$  or  $\int_0^T (\|\Lambda\theta\|_{L^\infty} + \|\nabla\theta\|_{L^\infty} + \|\nabla u\|_{L^\infty}) dt = \infty$ .

Next we give the proof of uniqueness. Let  $T > 0$  be the maximal existence time of the solution to the system (1.1), and assume that  $\theta_1, \theta_2 \in C([0, T^*]; H^s)$ ,  $T^* < T$ , are two solutions to (1.1) with velocities  $u_1 = \mathcal{R}\theta_1$  and  $u_2 = \mathcal{R}\theta - 2$ , respectively, and the same initial data  $\theta_0 \in H^s$ . Denote  $\theta = \theta_1 - \theta_2$  and  $u = u_1 - u_2$ , then we have

$$\partial_t \theta + u \cdot \nabla \theta_1 + u_2 \cdot \nabla \theta + \theta \operatorname{div} u_1 + \theta_2 \operatorname{div} u = -\nu \Lambda^\alpha \theta. \quad (2.10)$$

Multiplying both hand side of the equation (2.10) by  $\theta$ , we have

$$\frac{1}{2} \frac{d}{dt} \|\theta\|_{L^2}^2 + \int_{\mathbb{R}^N} u \cdot \nabla \theta_1 \theta dx + \int_{\mathbb{R}^N} u_2 \cdot \nabla \theta \theta dx + \int_{\mathbb{R}^N} \theta \Lambda \theta_1 \theta dx + \int_{\mathbb{R}^N} \theta_2 \Lambda \theta \theta dx = - \int_{\mathbb{R}^N} \nu \Lambda^\alpha \theta \theta dx. \quad (2.11)$$

We can calculate

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\theta\|_{L^2}^2 + \nu \|\Lambda^{\frac{\alpha}{2}} \theta\|_{L^2}^2 \\ & \leq - \int_{\mathbb{R}^N} u \cdot \nabla \theta_1 \theta dx - \frac{1}{2} \int_{\mathbb{R}^N} u_2 \cdot \nabla |\theta|^2 dx - \int_{\mathbb{R}^N} \theta \Lambda \theta_1 \theta dx - \frac{1}{2} \int_{\mathbb{R}^N} \theta_2 \Lambda \theta^2 dx \\ & \leq C(\|\nabla \theta_1\|_{L^\infty} \|\theta\|_{L^2}^2 + \|\nabla u_2\|_{L^\infty} \|\theta\|_{L^2}^2 + \|\Lambda \theta_1\|_{L^\infty} \|\theta\|_{L^2}^2 + \frac{1}{2} \|\Lambda \theta_2\|_{L^\infty} \|\theta\|_{L^2}^2) \\ & \leq C(\|\theta_1\|_{H^s} + \|u_2\|_{H^s} + \|\theta_1\|_{H^s} + \|\theta_2\|_{H^s}) \|\theta\|_{L^2}^2 \\ & \leq C(\|\theta_1\|_{H^s} + \|\theta_2\|_{H^s}) \|\theta\|_{L^2}^2. \end{aligned} \quad (2.12)$$

Here we use  $s > \frac{N}{2} + 1$  and  $\theta_2 \geq 0$ . Applying the Gronwall's inequality to the inequality (2.12) and using the fact that  $\|\theta_1(t)\|_{H^s}$  and  $\|\theta_2(t)\|_{H^s}$  is bounded for  $t \in [0, T^*]$ , we can obtain the desired uniqueness result.

**Proof of Theorem 1.3** We will prove Theorem 1.3 by using the fixed point principle by constructing contraction mapping.

We re-write the system (1.1) into the equivalent integral system

$$\theta(x, t) = G_\alpha(t) \theta_0(x) - \int_0^t G_\alpha(t - \tau) \operatorname{div}(u\theta)(\tau) d\tau, \quad (2.13)$$

where  $G_\alpha(t)$  is given by the Fourier transform  $\widehat{G_\alpha(t)} = e^{-\nu|\xi|^\alpha t}$ , and satisfies the following boundedness [29, 30, 26].

**Lemma 2.1** Assume  $1 \leq p \leq q \leq \infty$ . Then, for any  $t > 0$ , the operators  $G_\alpha(t)$  and  $\nabla G_\alpha(t)$  are bounded from  $L^p$  to  $L^q$ . Furthermore, we have, for any  $f \in L^p$ , that

$$\|G_\alpha(t)f\|_{L^q} \leq Ct^{-\frac{2}{\alpha}(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p}, \quad (2.14)$$

$$\|\nabla G_\alpha(t)f\|_{L^q} \leq Ct^{-\frac{1}{\alpha}-\frac{2}{\alpha}(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p}, \quad (2.15)$$

where  $C$  is a constant depending only on  $\alpha, p$  and  $q$ .

Further, assume that  $u$  and  $\theta$  are in  $L^q([0, T]; L^p(\mathbb{R}^N))$ , then the operator  $A(u, \theta) \equiv \int_0^t \nabla G_\alpha(t - \tau)(u\theta) d\tau$  is bounded in  $L^q([0, T]; L^p(\mathbb{R}^N))$  with

$$\|A(u, \theta)\|_{L^q([0, T]; L^p(\mathbb{R}^N))} \leq C\|u\|_{L^q([0, T]; L^p(\mathbb{R}^N))} \cdot \|\theta\|_{L^q([0, T]; L^p(\mathbb{R}^N))}, \quad (2.16)$$

where  $C$  is a constant depending only on  $\alpha, p$  and  $q$ .

For  $l = 0$ , define the space  $X = \{\theta \in L^q([0, T]; L^p) : \|\theta\|_X \leq M < \infty\}$  with the norm  $\|\cdot\|_X = \|\cdot\|_{L^q([0, T]; L^p)}$ , and define the mapping  $F$  mapping  $\theta \in X$  to  $F(\theta)$  by

$$F(\theta)(x, t) = G_\alpha(t)\theta_0(x) - \int_0^t G_\alpha(t - \tau) \operatorname{div}(u\theta)(\tau) d\tau \quad (2.17)$$

with the velocity  $u = \mathcal{R}\theta$ . In the following, we will prove that

- (i) If  $\theta \in X$ , then  $F(\theta) \in X$ ;
- (ii) For any  $\theta, \tilde{\theta} \in X$ , then  $\|F(\theta) - F(\tilde{\theta})\|_X \leq \frac{1}{2}\|\theta - \tilde{\theta}\|_X$  for some  $T > 0$ .

In fact, by using (2.14) in Lemma 2.1, we can easily conclude that  $F(0) = G_\alpha(t)\theta_0$  is bounded in  $L^q([0, T]; L^p(\mathbb{R}^N))$ , i.e.,

$$\begin{aligned} \|G_\alpha(t)\theta_0\|_{L^q([0, T]; L^p)} &= \left[ \int_0^T \|G_\alpha(t)\theta_0\|_{L^p}^q dt \right]^{\frac{1}{q}} \\ &\leq \left[ \int_0^T \|\theta_0\|_{L^p}^q dt \right]^{\frac{1}{q}} \\ &\leq \|\theta_0\|_{L^p} \left[ \int_0^T dt \right]^{\frac{1}{q}} \\ &\leq CT^{\frac{1}{q}} \|\theta_0\|_{L^p}. \end{aligned} \quad (2.18)$$

Now we choose  $M = 3CT^{\frac{1}{q}}\|\theta_0\|_{L^p}$  sufficiently small by using  $\alpha > 1$  and letting  $T$  sufficiently small, and hence we have  $\|F(0)\|_{L^q([0, T]; L^p)} = \|G_\alpha(t)\theta_0\|_{L^q([0, T]; L^p)} \leq \frac{M}{3}$ .

Let  $\theta$  and  $\tilde{\theta}$  be any two elements of  $X$ , where  $u$  and  $\tilde{u}$  be the velocities corresponding to  $\theta$  and  $\tilde{\theta}$ , respectively. Then, using (2.16) in Lemma 2.1, we have

$$\begin{aligned} &\|F(\theta) - F(\tilde{\theta})\|_{L^q([0, T]; L^p)} \\ &= \left\| \int_0^t \nabla G(t - \tau)(u\theta)(\tau) d\tau - \int_0^t \nabla G(t - \tau)(\tilde{u}\tilde{\theta})(\tau) d\tau \right\|_{L^q([0, T]; L^p)} \\ &= \|A(u, \theta) - A(\tilde{u}, \tilde{\theta})\|_{L^q([0, T]; L^p)} \\ &= \|A(u - \tilde{u}, \theta) + A(\tilde{u}, \theta - \tilde{\theta})\|_{L^q([0, T]; L^p)} \\ &\leq \|A(u - \tilde{u}, \theta)\|_{L^q([0, T]; L^p)} + \|A(\tilde{u}, \theta - \tilde{\theta})\|_{L^q([0, T]; L^p)} \\ &\leq C\|u - \tilde{u}\|_{L^q([0, T]; L^p)}\|\theta\|_{L^q([0, T]; L^p)} + C\|\tilde{u}\|_{L^q([0, T]; L^p)}\|\theta - \tilde{\theta}\|_{L^q([0, T]; L^p)}. \end{aligned} \quad (2.19)$$

Because  $u$  and  $\tilde{u}$  are Riesz transforms of  $\theta$  and  $\tilde{\theta}$ , respectively, the classical Calderon-Zygmund singular integral estimates imply that

$$\|u\|_{L^q([0, T]; L^p)} \leq C\|\theta\|_{L^q([0, T]; L^p)}, \quad (2.20)$$

and

$$\|\tilde{u}\|_{L^q([0, T]; L^p)} \leq C\|\tilde{\theta}\|_{L^q([0, T]; L^p)}. \quad (2.21)$$

Substituting inequalities (2.20) and (2.21) into (2.19), we get

$$\begin{aligned} \|F(\theta) - F(\tilde{\theta})\|_{L^q([0, T]; L^p)} &\leq C(\|\theta\|_{L^q([0, T]; L^p)} + \|\tilde{\theta}\|_{L^q([0, T]; L^p)})\|\theta - \tilde{\theta}\|_{L^q([0, T]; L^p)} \\ &\leq CM\|\theta - \tilde{\theta}\|_{L^q([0, T]; L^p)}. \end{aligned} \quad (2.22)$$

Hence, using (2.22) and letting  $M$  to be small enough, we have

$$\begin{aligned}
\|F(\theta)\|_{L^q([0,T];L^p)} &= \|F(\theta) - F(0) + F(0)\|_{L^q([0,T];L^p)} \\
&\leq \|F(\theta) - F(0)\|_{L^q([0,T];L^p)} + \|F(0)\|_{L^q([0,T];L^p)} \\
&\leq CM\|\theta\|_{L^q([0,T];L^p)} + \frac{M}{3} \\
&\leq CM^2 + \frac{M}{3} \\
&\leq M
\end{aligned}$$

and

$$\|F(\theta) - F(\tilde{\theta})\|_{L^q([0,T];L^p)} \leq \frac{1}{2}(\|\theta - \tilde{\theta}\|_{L^q([0,T];L^p)}).$$

By the contracting mapping principle, there exists a unique function  $\theta \in X$  such that  $F(\theta) = \theta$  and, hence, there exists a time  $T > 0$  such that the system (1.1) has a solution  $\theta \in L^q([0, T]; L^p)$ .

For  $l > 0$ , define the space  $X = \{\theta \in L^q([0, T]; W^{l,p}) : \|\theta\|_X \leq M < \infty\}$  with the norm  $\|\cdot\|_X = \|\cdot\|_{L^q([0,T];W^{l,p})}$ . Similar to the proof of the case  $l = 0$ , we can obtain the local existence of the smooth solution to the system (1.1).

This ends the proof of Theorem 1.3.

### §3 Global existence of strong and smooth solution: proofs of Theorems 1.4 and 1.5

In this section, we will prove the global existence of strong or smooth solution to the system (1.1) for the sub-critical and critical cases  $1 \leq \alpha \leq 2$  by the careful energy methods.

**Proof of Theorem 1.4:** If we assume that  $s > \frac{N}{2} + 1$ , then the local existence can be guaranteed by Theorem 1.1. For general  $s > 0$ , we can prove the local existence of the strong or smooth solution by the fixed point theory as in the proof of Theorem 1.3. To prove the global existence, it suffices to establish the a priori estimates globally in time. This is divided into the following three steps.

#### Step 1: $L^p$ -estimate and Maximum principle

When  $\alpha = 2$ , the result is obvious. We only need consider the case  $1 < \alpha < 2$ .

We notice the fact that, if  $\theta_0(x) \geq 0$ , then  $\theta(x, t) \geq 0$  in  $\overline{\Omega_T} = \mathbb{R}^N \times (0, T]$ .

Multiplying both sides of equation (1.1)<sub>1</sub> by  $\theta^p(x, t)$  and integrating the resulting equation in  $\mathbb{R}^N$ , one get

$$\begin{aligned}
\frac{1}{p+1} \frac{d}{dt} \int_{\mathbb{R}^N} \theta^{p+1} dx &= \int_{\mathbb{R}^N} -\operatorname{div}(R(\theta)\theta)\theta^p dx - \nu \int_{\mathbb{R}^N} (-\Delta)^{\frac{\alpha}{2}} \theta \cdot \theta^p dx \\
&= \int_{\mathbb{R}^N} R(\theta)\theta \cdot p\theta^{p-1} \nabla \theta dx - \nu \int_{\mathbb{R}^N} \Lambda^\alpha \theta \cdot \theta^p dx \\
&= \frac{p}{p+1} \int_{\mathbb{R}^N} R(\theta) \nabla \theta^{p+1} dx - \nu \int_{\mathbb{R}^N} \theta^p \Lambda^\alpha \theta dx \\
&= -\frac{p}{p+1} \int_{\mathbb{R}^N} \theta^{p+1} \Lambda \theta dx - \nu \int_{\mathbb{R}^N} \theta^p \Lambda^\alpha \theta dx \leq 0
\end{aligned}$$

with the aid of (1.14) in Lemma 1.12. Hence, we have

$$\int_{\mathbb{R}^N} \theta^{p+1}(x) dx \leq \int_{\mathbb{R}^N} \theta_0^{p+1}(x) dx, \quad \forall p > 1,$$

i.e.,

$$\|\theta\|_{L^p} \leq \|\theta_0\|_{L^p}.$$

In particular, if we take  $p \rightarrow \infty$ , we get

$$\|\theta\|_{L^\infty} \leq \|\theta_0\|_{L^\infty}.$$

**Step 2 : A priori estimate in  $H^s$  for the case  $s = 2$**

Multiplying both sides of equation (1.1)<sub>1</sub> by  $\Lambda^4 \theta$  and taking the inner product with the resulting equation in  $L^2$ , we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Lambda^2 \theta\|_{L^2}^2 &= - \int_{\mathbb{R}^N} \Lambda^{2+\frac{\alpha}{2}} \theta \Lambda^{2-\frac{\alpha}{2}} \operatorname{div}(u\theta) dx - \nu \|\Lambda^{2+\frac{\alpha}{2}} \theta\|_{L^2}^2 \\ &\leq \|\Lambda^{2+\frac{\alpha}{2}} \theta\|_{L^2} \|\Lambda^{2+1-\frac{\alpha}{2}}(u\theta)\|_{L^2} - \nu \|\Lambda^{2+\frac{\alpha}{2}} \theta\|_{L^2}^2, \end{aligned} \quad (3.1)$$

where we have used the Hölder inequality and the calculus inequality  $\|\Lambda^{2-\frac{\alpha}{2}} \operatorname{div}(u\theta)\|_{L^2} \leq \|\Lambda^{2+1-\frac{\alpha}{2}}(u\theta)\|_{L^2}$ .

Using the inequalities for the Calderon-Zygmund type singular integrals on  $u = \mathcal{R}\theta$ , we have

$$\|u\|_{L^p} \leq C \|\theta\|_{L^p}, \quad \|\Lambda^{3-\frac{\alpha}{2}} u\|_{L^q} \leq C \|\Lambda^{3-\frac{\alpha}{2}} \theta\|_{L^q}, \quad 1 < p, q < +\infty. \quad (3.2)$$

By using (1.10) in Lemma 1.9 and (3.2), we have

$$\begin{aligned} \|\Lambda^{2+1-\frac{\alpha}{2}}(u\theta)\|_{L^2} &\leq C(\|u\|_{L^p} \|\Lambda^{3-\frac{\alpha}{2}} \theta\|_{L^q} + \|\theta\|_{L^p} \|\Lambda^{3-\frac{\alpha}{2}} u\|_{L^q}) \\ &\leq C(\|\theta\|_{L^p} \|\Lambda^{3-\frac{\alpha}{2}} \theta\|_{L^q} + \|\theta\|_{L^p} \|\Lambda^{3-\frac{\alpha}{2}} \theta\|_{L^q}) \\ &\leq C \|\theta\|_{L^p} \|\Lambda^{3-\frac{\alpha}{2}} \theta\|_{L^q}, \end{aligned} \quad (3.3)$$

where  $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}, p, q > 2$ .

Putting (3.3) into (3.1), we have

$$\frac{1}{2} \frac{d}{dt} \|\Lambda^2 \theta\|_{L^2}^2 \leq C \|\theta\|_{L^p} \|\Lambda^{2+\frac{\alpha}{2}} \theta\|_{L^2} \|\Lambda^{3-\frac{\alpha}{2}} \theta\|_{L^q} - \nu \|\Lambda^{2+\frac{\alpha}{2}} \theta\|_{L^2}^2. \quad (3.4)$$

Using the Lemma 1.10, we have

$$\|\Lambda^{3-\frac{\alpha}{2}} \theta\|_{L^q} \leq C \|\Lambda^{3-\frac{\alpha}{2}+\delta} \theta\|_{L^2}, \quad (3.5)$$

where  $\frac{1}{q} = \frac{1}{2} - \frac{\delta}{N}$ .

Now we take  $\delta = \frac{N}{p}, p > \frac{N}{\alpha-1} \geq 2(1 < \alpha \leq 2)$ , and therefore  $1 + \delta < \alpha$ . Then apply the fractional type Gagliardo-Nirenberg inequality

$$\|\Lambda^{3-\frac{\alpha}{2}+\delta} \theta\|_{L^2} \leq \|\Lambda^{2+\frac{\alpha}{2}} \theta\|_{L^2}^a \|\Lambda^2 \theta\|_{L^2}^{1-a} \quad (3.6)$$

with the parameter  $a = \frac{2-\alpha+2\delta}{\alpha} < 1$ .

Putting (3.4), (3.5) and (3.6) together, and using the Young's inequality, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Lambda^2 \theta\|_{L^2}^2 &\leq C \|\theta\|_{L^p} \|\Lambda^{2+\frac{\alpha}{2}} \theta\|_{L^2}^{a+1} \|\Lambda^2 \theta\|_{L^2}^{1-a} - \nu \|\Lambda^{2+\frac{\alpha}{2}} \theta\|_{L^2}^2 \\ &\leq C \|\theta\|_{L^p}^{\frac{2}{1-a}} \|\Lambda^2 \theta\|_{L^2}^2 - \frac{\nu}{2} \|\Lambda^{2+\frac{\alpha}{2}} \theta\|_{L^2}^2, \end{aligned}$$

which, together with  $L^p$  estimate of  $\theta$  in Step 1, gives

$$\frac{1}{2} \frac{d}{dt} \|\Lambda^2 \theta\|_{L^2}^2 \leq C(\nu, \|\theta_0\|_{L^p}) \|\Lambda^2 \theta\|_{L^2}^2,$$

which gives

$$\|\Lambda^2 \theta\|_{L^2}(t) \leq \|\Lambda^2 \theta_0\|_{L^2} e^{Ct}. \quad (3.7)$$

**Step 3: A priori estimate in  $H^s$  for the case  $s > 0$**

Multiplying both sides of equation (1.1)<sub>1</sub> by  $\Lambda^{2s} \theta$  and taking the inner product with the resulting equation in  $L^2$ , we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Lambda^s \theta\|_{L^2}^2 &= - \int_{\mathbb{R}^N} \Lambda^{s+\frac{\alpha}{2}} \theta \Lambda^{s-\frac{\alpha}{2}} \operatorname{div}(u\theta) dx - \nu \|\Lambda^{s+\frac{\alpha}{2}} \theta\|_{L^2}^2 \\ &\leq \|\Lambda^{s+\frac{\alpha}{2}} \theta\|_{L^2} \|\Lambda^{s+1-\frac{\alpha}{2}}(u\theta)\|_{L^2} - \nu \|\Lambda^{s+\frac{\alpha}{2}} \theta\|_{L^2}^2, \end{aligned} \quad (3.8)$$

where we have used the Hölder inequality and the calculus inequality  $\|\Lambda^{s-\frac{\alpha}{2}} \operatorname{div}(u\theta)\|_{L^2} \leq \|\Lambda^{s+1-\frac{\alpha}{2}}(u\theta)\|_{L^2}$ .

Using the inequalities for the Calderon-Zygmund type singular integrals  $\mathcal{R}$

$$\|u\|_{L^p} \leq C \|\theta\|_{L^p}, \quad \|\Lambda^{s+1-\frac{\alpha}{2}} u\|_{L^q} \leq C \|\Lambda^{s+1-\frac{\alpha}{2}} \theta\|_{L^q}, \quad 1 < p, q < \infty$$

and Lemma 1.9, we have

$$\begin{aligned} \|\Lambda^{s+1-\frac{\alpha}{2}}(u\theta)\|_{L^2} &\leq C(\|u\|_{L^p} \|\Lambda^{s+1-\frac{\alpha}{2}} \theta\|_{L^q} + \|\theta\|_{L^p} \|\Lambda^{s+1-\frac{\alpha}{2}} u\|_{L^q}) \\ &\leq C(\|\theta\|_{L^p} \|\Lambda^{s+1-\frac{\alpha}{2}} \theta\|_{L^q} + \|\theta\|_{L^p} \|\Lambda^{s+1-\frac{\alpha}{2}} \theta\|_{L^q}) \\ &\leq C(\|\theta\|_{L^p} \|\Lambda^{s+1-\frac{\alpha}{2}} \theta\|_{L^q}), \end{aligned} \quad (3.9)$$

where  $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}, p, q > 2$ .

Putting (3.9) into (3.8), we have

$$\frac{1}{2} \frac{d}{dt} \|\Lambda^s \theta\|_{L^2}^2 \leq C \|\theta\|_{L^p} \|\Lambda^{s+\frac{\alpha}{2}} \theta\|_{L^2} \|\Lambda^{s+1-\frac{\alpha}{2}} \theta\|_{L^q} - \nu \|\Lambda^{s+\frac{\alpha}{2}} \theta\|_{L^2}^2. \quad (3.10)$$

Using the Lemma 1.10, we have

$$\|\Lambda^{s+1-\frac{\alpha}{2}} \theta\|_{L^q} \leq C \|\Lambda^{s+1-\frac{\alpha}{2}+\delta} \theta\|_{L^2}, \quad (3.11)$$

where  $\frac{1}{q} = \frac{1}{2} - \frac{\delta}{N}$ .

Now we take  $\delta = \frac{N}{p}, p > \frac{N}{\alpha-1} \geq 2(1 < \alpha \leq 2)$ , and therefore  $1 + \delta < \alpha$ . Then apply the fractional type Gagliardo-Nirenberg inequality

$$\|\Lambda^{s+1-\frac{\alpha}{2}+\delta} \theta\|_{L^2} \leq \|\Lambda^{s+\frac{\alpha}{2}} \theta\|_{L^2}^a \|\Lambda^s \theta\|_{L^2}^{1-a} \quad (3.12)$$

with the parameter  $a = \frac{2-\alpha+2\delta}{\alpha} < 1$ .

Putting (3.10), (3.11) and (3.12) together, and using the Young's inequality, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Lambda^s \theta\|_{L^2}^2 &\leq C \|\theta\|_{L^p} \|\Lambda^{s+\frac{\alpha}{2}} \theta\|_{L^2}^{a+1} \|\Lambda^s \theta\|_{L^2}^{1-a} - \nu \|\Lambda^{s+\frac{\alpha}{2}} \theta\|_{L^2}^2 \\ &\leq C \|\theta\|_{L^p}^{\frac{2}{1-a}} \|\Lambda^s \theta\|_{L^2}^2 - \frac{\nu}{2} \|\Lambda^{s+\frac{\alpha}{2}} \theta\|_{L^2}^2, \end{aligned}$$

which, together with  $L^p$  estimate on  $\theta$  in Step 1, gives

$$\frac{1}{2} \frac{d}{dt} \|\Lambda^s \theta\|_{L^2}^2 \leq C(\nu, \|\theta_0\|_{L^p}) \|\Lambda^s \theta\|_{L^2}^2,$$

which gives

$$\|\Lambda^s \theta\|_{L^2}(t) \leq \|\Lambda^s \theta_0\|_{L^2} e^{Ct}. \quad (3.13)$$

Using the a priori estimates (3.7), (3.13) and the standard extension argument we can conclude the global existence result. The proof of Theorem 1.4 is complete.

**Proof of Theorem 1.5** The proof is analogous to the critical dissipative quasi-geostrophic

equation that is shown in [5]. We give two key points of the proof. First, we have

$$\begin{aligned} & \int_{\mathbb{R}^N} \theta_\lambda^2(t_2, x) dx + 2 \int_{t_1}^{t_2} \int_{\mathbb{R}^N} |\Lambda^{\frac{1}{2}} \theta_\lambda|^2 dx dt + 2 \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \operatorname{div}(u \theta_\lambda) \theta_\lambda dx dt \\ & \leq \int_{\mathbb{R}^N} \theta_\lambda^2(t_1, x) dx, 0 < t_1 < t_2. \end{aligned} \quad (3.14)$$

Next, we only need to show the term  $\int_{t_1}^{t_2} \int_{\mathbb{R}^N} \operatorname{div}(u \theta_\lambda) \theta_\lambda dx dt$  is positive. In fact, by the direct calculation, we have

$$\begin{aligned} \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \operatorname{div}(u \theta_\lambda) \theta_\lambda dx dt &= \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \operatorname{div} u \cdot \theta_\lambda^2 dx dt + \int_{t_1}^{t_2} \int_{\mathbb{R}^N} u \cdot \nabla \theta_\lambda \cdot \theta_\lambda dx dt \\ &= \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \operatorname{div} u \cdot \theta_\lambda^2 dx dt - \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \operatorname{div} u \cdot \frac{|\theta_\lambda|^2}{2} dx dt \\ &= \frac{1}{2} \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \Lambda \theta_\lambda \cdot \theta_\lambda^2 dx dt \geq 0. \end{aligned} \quad (3.15)$$

Here we have used the relationship  $u = \mathcal{R}\theta$  and  $\operatorname{div} u = \operatorname{div} \mathcal{R}\theta = \Lambda\theta$ . Combining (3.14) and (3.15), we obtain (1.5). Then utilize the same strategy as [5, 6] to finish the proof of Theorem 1.5.

## §4 Global existence of the weak solution: proofs of Theorems 1.6 and 1.7

In this section, we will prove Theorems 1.6 and 1.7 by employing the vanishing viscosity method used in [14, 9]. We consider the general case  $0 < \alpha \leq 2$ .

**Definition 4.1.** A solution  $\theta(x, t)$  is called the weak solution to system (1.1), if for any smooth function  $\phi \in C_0^\infty([0, \tau] \times \mathbb{R}^N)$ , it satisfies

$$\begin{aligned} & \int_{\mathbb{R}^N} \theta(x, t) \phi(x, t) dx - \int_{\mathbb{R}^N} \theta_0(x) \phi(x, 0) dx + \int_0^\tau \int_{\mathbb{R}^N} [-\theta(x, t) \partial_t \phi(x, t) \\ & - u \theta \cdot \nabla \phi(x, t) + \nu \theta(x, t) \Lambda^\alpha \phi(x, t)] dx dt = 0, \end{aligned}$$

where the velocity  $u = \mathcal{R}\theta$ .

Let  $\varepsilon > 0$  be a small parameter and we will approximate problem (1.1) by considering the regularized system of (1.1) with a small viscosity term

$$\begin{cases} \frac{\partial \theta_\varepsilon}{\partial t} + \operatorname{div}(u_\varepsilon \theta_\varepsilon) + \nu \Lambda^\alpha \theta_\varepsilon = \varepsilon \Delta \theta_\varepsilon, \\ u_\varepsilon = \mathcal{R} \theta_\varepsilon, \\ \theta_\varepsilon(x, 0) = \theta_0^\varepsilon. \end{cases} \quad (4.1)$$

for  $0 < \varepsilon \leq 1$ ,  $\theta_0^\varepsilon = \psi_\varepsilon * \theta_0$ ,  $\psi_\varepsilon(x) = \varepsilon^{-N} \psi(\frac{x}{\varepsilon})$  and  $\psi$  satisfying

$$\psi \geq 0, \quad \psi \in C_0^\infty(\mathbb{R}^N) \quad \text{and} \quad \|\psi\|_{L^1} = 1.$$

For any fixed  $\varepsilon > 0$ , by the standard parabolic theory, as in the proof of Theorem 1.1, we can prove the following global existence results on the smooth solution to the regularized system (4.1).

**Proposition 4.1** *For any  $\varepsilon > 0$  and for any  $\tau > 0$ , there exists a unique solution  $\theta_\varepsilon$  of (4.1)*

satisfying  $\theta_\varepsilon \in C([0, \tau]; H^s(\mathbb{R}^N))$  ( $s > \frac{N}{2} + 1$ ). Moreover, if  $\theta_0 \geq 0$ , then  $\theta^\varepsilon(x, t) \geq 0$ .

We want to establish the a priori estimates for  $\theta^\varepsilon$  with respect to  $\varepsilon$ , and then to perform the limit  $\lim_{\varepsilon \rightarrow 0} \theta^\varepsilon = \theta$  in the sense of weak convergence, and to verify that the limit function  $\theta$  is a weak solution of the system (1.1) in the sense of Definition 4.1.

We multiply both sides of equations (4.1)<sub>1</sub> by  $\theta_\varepsilon$  to get

$$\frac{1}{2} \frac{d}{dt} \|\theta_\varepsilon\|_{L^2}^2 + \nu \|\Lambda^{\frac{\alpha}{2}} \theta_\varepsilon\|_{L^2}^2 + \varepsilon \|\nabla \theta_\varepsilon\|_{L^2}^2 \leq \int_{\mathbb{R}^N} \mathcal{R}(\theta_\varepsilon) \theta_\varepsilon \cdot \nabla \theta_\varepsilon dx = -\frac{1}{2} \int_{\mathbb{R}^N} \theta_\varepsilon^2 \Lambda \theta_\varepsilon \leq 0, \quad (4.2)$$

where we have used (1.14) in Lemma 1.12 and  $\theta \geq 0$ .

Then we integrate (4.2) in time to get

$$\|\theta_\varepsilon(\tau)\|_{L^2}^2 + 2\nu \int_0^\tau \|\Lambda^{\frac{\alpha}{2}} \theta_\varepsilon(s)\|_{L^2}^2 ds \leq \|\theta_0\|_{L^2}^2, \quad \forall \tau. \quad (4.3)$$

In particular, we obtain

$$\theta_\varepsilon \in C([0, \tau]; L^2(\mathbb{R}^N)), \quad \sup_{0 \leq t \leq \tau} \|\theta_\varepsilon\|_{L^2(\mathbb{R}^N)} \leq \|\theta_0\|_{L^2(\mathbb{R}^N)}, \text{ and } \max_{0 \leq t \leq \tau} \|\theta_\varepsilon(t)\|_{L^2}^2 \leq \|\theta_0\|_{L^2}^2. \quad (4.4)$$

Using  $u^\varepsilon = \mathcal{R}^\varepsilon$  and  $L^2$  boundedness of the Riesz transform, one get

$$\|u_\varepsilon\|_{L^2(\mathbb{R}^N)} \leq C \|\theta_\varepsilon\|_{L^2(\mathbb{R}^N)} \leq C \|\theta_0\|_{L^2(\mathbb{R}^N)}. \quad (4.5)$$

Next we pass to the limit  $\varepsilon \rightarrow 0$  in (4.1) by using the Aubin-Lions compactness lemma.

First of all, by the previous a priori estimate as in (4.4), we obtain  $\theta_\varepsilon \in C([0, \tau]; L^2(\mathbb{R}^N))$  and

$$\max\{\|\theta_\varepsilon(t)\|_{L^2} : 0 \leq t \leq \tau\} \leq M < \infty. \quad (4.6)$$

Secondly, we want to prove that, for any  $\phi \in C_0^\infty(\mathcal{R}^N)$ ,  $\{\phi \theta_\varepsilon\}$  is uniformly Lipschitz in the interval of time  $[0, \tau]$  with respect to the space  $H^{-p}$  with  $p > \frac{N}{2} + 2$ , i.e.

$$\|\phi \theta_\varepsilon(t_2) - \phi \theta_\varepsilon(t_1)\|_{H^{-p}} \leq C |t_2 - t_1|, \quad 0 \leq t_1, t_2 \leq \tau \quad (4.7)$$

for some positive constant  $C > 0$ .

Because  $\theta_\varepsilon$  is a strong solution of (4.1) and is continuous, it follows that

$$\|\phi \theta_\varepsilon(t_2) - \phi \theta_\varepsilon(t_1)\|_{H^{-p}} = \left\| \int_{t_1}^{t_2} \phi \frac{d}{dt} \theta_\varepsilon dt \right\|_{H^{-p}} \leq \max_{t_1 \leq t \leq t_2} \{M(t)\} (t_2 - t_1), \quad (4.8)$$

where

$$M(t) = \|\operatorname{div}(\phi \mathcal{R}(\theta_\varepsilon) \theta_\varepsilon)\|_{H^{-p}} + \|\nabla \phi \mathcal{R}(\theta_\varepsilon) \theta_\varepsilon\|_{H^{-p}} + \nu \|\phi \Lambda^\alpha \theta_\varepsilon\|_{H^{-p}} + \varepsilon \|\phi \Delta \theta_\varepsilon\|_{H^{-p}}.$$

Using the Sobolev's imbedding theorem and using (4.4) and (??), we have

$$\begin{aligned} \|\nabla \phi \mathcal{R}(\theta_\varepsilon) \theta_\varepsilon\|_{H^{-p}} &\leq C(p) \|\widehat{\nabla \phi \mathcal{R}(\theta_\varepsilon) \theta_\varepsilon}\|_{L^\infty} \\ &\leq C(p) \|\nabla \phi\|_{L^\infty} \|\mathcal{R}(\theta_\varepsilon) \theta_\varepsilon\|_{L^1} \\ &\leq C(p, \phi) \|\theta_\varepsilon\|_{L^2}^2 \\ &\leq C(p, \phi) \|\theta_0\|_{L^2}^2. \end{aligned} \quad (4.9)$$

Similarly, we have

$$\begin{aligned} \|\operatorname{div}((\phi \mathcal{R}(\theta_\varepsilon) \theta_\varepsilon))\|_{H^{-p}} &\leq \|\phi \mathcal{R}(\theta_\varepsilon) \theta_\varepsilon\|_{H^{1-p}} \\ &\leq C(\phi, p) \|\theta_\varepsilon\|_{L^2}^2 \\ &\leq C(\phi, p) \|\theta_0\|_{L^2}^2. \end{aligned} \quad (4.10)$$

Applying the convolution property of the Fourier transforms, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^N} \widehat{\phi(y)} |\xi - y|^\alpha \widehat{\theta_\varepsilon}(\xi - y) dy \right| &\leq C \int_{\mathbb{R}^N} (|\xi|^\alpha + |y|^\alpha) |\widehat{\phi}(y)| |\widehat{\theta_\varepsilon}(\xi - y)| dy \\ &\leq C(1 + |\xi|^\alpha) \|\phi\|_{H^\alpha} \|\theta_\varepsilon(0)\|_{L^2}, \end{aligned}$$

which gives

$$\begin{aligned} \|\phi \Lambda^\alpha \theta_\varepsilon\|_{H^{-p}} &\leq C(\phi) \|\theta_\varepsilon\|_{L^2} \left( \int_{\mathbb{R}^N} \frac{(1 + |\xi|^\alpha)^2}{(1 + |\xi|^2)^p} d\xi \right)^{\frac{1}{2}} \\ &\leq C(p, \phi) \|\theta_\varepsilon\|_{L^2} \leq C(p, \phi) \|\theta_0\|_{L^2}. \end{aligned} \quad (4.11)$$

Similarly, we obtain

$$\|\phi \Delta \theta_\varepsilon\|_{H^{-p}} \leq C(p, \phi) \|\theta_0\|_{L^2}. \quad (4.12)$$

Putting (4.8) together with (4.9)-(4.12), we obtain (4.7).

From (4.6)-(4.7), conditions (i) and (ii) of the Aubin-Lions lemma [9] are satisfied. Therefore, there exists a subsequence and a function  $\theta \in C([0, \tau]; L^2(\mathbb{R}^N))$  such that

$$\theta_\varepsilon \rightharpoonup \theta \quad \text{in } L^2(\mathbb{R}^N) \quad \text{a.e. t and } \max_{0 \leq t \leq \tau} \|\phi \theta_\varepsilon(t) - \phi \theta(t)\|_{H^{-p}} \rightarrow 0. \quad (4.13)$$

We take the limit in the weak formulation of the problem (4.1)

$$\begin{aligned} \int_{\mathbb{R}^N} \theta_\varepsilon(x, \tau) \phi dx - \int_{\mathbb{R}^N} \theta_\varepsilon(x, 0) \phi(x, 0) dx + \int_0^\tau \int_{\mathbb{R}^N} [-\theta_\varepsilon(x, t) \partial_t \phi(x, t) - u_\varepsilon \theta_\varepsilon \cdot \nabla \phi(x, t) \\ + \nu \theta_\varepsilon(x, t) \Lambda^\alpha \phi(x, t) - \varepsilon \theta_\varepsilon(x, t) \Delta \phi(x, t)] dx dt = 0, \end{aligned}$$

and let  $\varepsilon \rightarrow 0$ , we get

$$\begin{aligned} \int_{\mathbb{R}^N} \theta(x, \tau) \phi(x, t) dx - \int_{\mathbb{R}^N} \theta_0(x, t) \phi(x, 0) + \int_0^\tau \int_{\mathbb{R}^N} [\theta(x, t) \partial_t \phi(x, t) + \nu \theta(x, t) \Lambda^\alpha \phi(x, t)] dx dt \\ + \lim_{\varepsilon \rightarrow 0} \int_0^\tau \int_{\mathbb{R}^N} \theta_\varepsilon u_\varepsilon \cdot \nabla \phi(x, t) dx dt = 0, \end{aligned} \quad (4.14)$$

Now we rewrite the last term in the left hand side of (4.14) as follows:

$$\begin{aligned} &\int_0^\tau \int_{\mathbb{R}^N} \theta_\varepsilon u_\varepsilon \cdot \nabla \phi(x, t) dx dt \\ &\leq \int_0^\tau \int_{\mathbb{R}^N} (\theta_\varepsilon - \theta) u_\varepsilon \cdot \nabla \phi dx dt + \int_0^\tau \int_{\mathbb{R}^N} \theta u_\varepsilon \cdot \nabla \phi dx dt \\ &\equiv I_1 + I_2. \end{aligned} \quad (4.15)$$

The first term  $I_1$  can be estimated by

$$\begin{aligned} |I_1| &= \left| \int_0^\tau \int_{\mathbb{R}^N} (\theta_\varepsilon - \theta) u_\varepsilon \cdot \nabla \phi dx dt \right| \\ &\leq \int_0^\tau \|u_\varepsilon\|_{H^{\frac{\alpha}{2}}} \|(\theta_\varepsilon - \theta) \nabla \phi\|_{H^{-\frac{\alpha}{2}}} dt \\ &\leq \max_{0 \leq t \leq \tau} \|(\theta_\varepsilon - \theta) \nabla \phi\|_{H^{-\frac{\alpha}{2}}} \int_0^\tau (\|u_\varepsilon\|_{L^2} + \|\Lambda^{\frac{\alpha}{2}} u_\varepsilon\|_{L^2}) dt \\ &\leq C(\tau) \|\theta_0\|_{L^2} \max_{0 \leq t \leq T} \|(\theta_\varepsilon - \theta) \cdot \nabla \phi\|_{H^{-\frac{\alpha}{2}}} \rightarrow 0, \end{aligned} \quad (4.16)$$

where we have used the fact (4.3) and (4.13).

Then, by (4.15), (4.13) and (4.16), we have

$$\lim_{\varepsilon \rightarrow 0} \int_0^\tau \int_{\mathbb{R}^N} u_\varepsilon \theta_\varepsilon \cdot \nabla \phi dx dt = \int_0^\tau \int_{\mathbb{R}^N} u \theta \cdot \nabla \phi dx dt,$$



which, together with (4.14), completes the proof of Theorem 1.5.

**Proof of Theorem 1.7** Let  $\theta_1$  and  $\theta_2$  be two solutions to the system (1.1) with the velocities  $u_1$  and  $u_2$ , respectively. The difference  $\theta = \theta_1 - \theta_2$  satisfies

$$\partial_t \theta + \operatorname{div}(u_1 \theta) + \operatorname{div}(u \theta_2) + \nu \Lambda^\alpha \theta = 0, \quad (4.17)$$

where  $u = u_1 - u_2$ . Clearly,  $\theta(x, 0) = 0$ .

Now multiply both sides of (4.17) by  $\Lambda^{-1} \theta$  and integrate by parts, one get

$$\frac{d}{dt} \|\Lambda^{-\frac{1}{2}} \theta\|_{L^2}^2 + \nu \|\Lambda^{-\frac{1}{2}} (\Lambda^{\frac{\alpha}{2}} \theta)\|_{L^2}^2 \leq \left| \int_{\mathbb{R}^N} (u_1 \theta) \cdot (\nabla(\Lambda^{-1} \theta)) dx \right| + \left| \int_{\mathbb{R}^N} (u \theta_2) \cdot (\nabla(\Lambda^{-1} \theta)) dx \right|. \quad (4.18)$$

By the Hölder inequality, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^N} (u_1 \theta) \cdot (\nabla(\Lambda^{-1} \theta)) dx \right| &\leq \|u_1\|_{L^q} \|\theta\|_{L^p} \|\nabla(\Lambda^{-1} \theta)\|_{L^p} \\ &\leq \|\theta_1\|_{L^q} \|\theta\|_{L^p} \|\nabla(\Lambda^{-1} \theta)\|_{L^p}, \end{aligned} \quad (4.19)$$

and

$$\begin{aligned} \left| \int_{\mathbb{R}^N} (u \theta_2) \cdot (\nabla(\Lambda^{-1} \theta)) dx \right| &\leq \|u\|_{L^p} \|\theta_2\|_{L^q} \|\nabla(\Lambda^{-1} \theta)\|_{L^p} \\ &\leq \|\theta_2\|_{L^q} \|\theta\|_{L^p} \|\nabla(\Lambda^{-1} \theta)\|_{L^p}, \end{aligned} \quad (4.20)$$

where  $\frac{1}{q} + \frac{2}{p} = 1$ .

Thanks to the fact  $\widehat{\nabla(\Lambda^{-1})} = (\widehat{\mathcal{R}_1}, \widehat{\mathcal{R}_2}, \dots, \widehat{\mathcal{R}_N})$ , we get from (4.19) and (4.20) that

$$\left| \int_{\mathbb{R}^N} (u_1 \theta) \cdot (\nabla(\Lambda^{-1} \theta)) dx \right| + \left| \int_{\mathbb{R}^N} (u \theta_2) \cdot (\nabla(\Lambda^{-1} \theta)) dx \right| \leq C(\|\theta_1\|_{L^q} + \|\theta_2\|_{L^q}) \|\theta\|_{L^p}^2, \quad (4.21)$$

Using (1.12) in Lemma 1.10, one get

$$\|\theta\|_{L^p} \leq C \|\Lambda^{\frac{N}{2q}} \theta\|_{L^2} = C \|\Lambda^{\frac{N}{2q} + \frac{1}{2}} (\Lambda^{-\frac{1}{2}} \theta)\|_{L^2} \leq \|\Lambda^{-\frac{1}{2}} \theta\|_{L^2}^r \|\Lambda^{\frac{\alpha}{2}} (\Lambda^{-\frac{1}{2}} \theta)\|_{L^2}^{1-r}. \quad (4.22)$$

In the last inequality, we have used fractional type Gagliardo-Nirenberg inequality with  $\frac{\alpha-1}{N} - \frac{1}{q} = \frac{\alpha r}{N}$ .

Substituting (4.21) and (4.22) into (4.18), we conclude

$$\begin{aligned} &\frac{d}{dt} \|\Lambda^{-\frac{1}{2}} \theta\|_{L^2}^2 + \nu \|\Lambda^{-\frac{1}{2}} (\Lambda^{\frac{\alpha}{2}} \theta)\|_{L^2}^2 \\ &\leq C(\|\theta_1\|_{L^q} + \|\theta_2\|_{L^q}) \|\Lambda^{-\frac{1}{2}} \theta\|_{L^2}^{2r} \|\Lambda^{\frac{\alpha}{2}} (\Lambda^{-\frac{1}{2}} \theta)\|_{L^2}^{2(1-r)} \\ &\leq \frac{C}{\nu} (\|\theta_1\|_{L^q} + \|\theta_2\|_{L^q})^{\frac{1}{r}} \|\Lambda^{-\frac{1}{2}} \theta\|_{L^2}^2 + \frac{\nu}{2} \|\Lambda^{\frac{\alpha}{2}} (\Lambda^{-\frac{1}{2}} \theta)\|_{L^2}^2, \end{aligned}$$

i.e.,

$$\frac{d}{dt} \|\Lambda^{-\frac{1}{2}} \theta\|_{L^2}^2 \leq \frac{C}{\nu} (\|\theta_1\|_{L^q} + \|\theta_2\|_{L^q})^{\frac{1}{r}} \|\Lambda^{-\frac{1}{2}} \theta\|_{L^2}^2. \quad (4.23)$$

The Gronwall inequality implies that  $\theta = 0$  and we complete the proof.

**Remark 4.2** Furthermore, when  $\nu = 0$ , if a compactly supported initial condition  $\theta_0 \leq 0, \neq 0$  has a sufficiently big integral  $M = -\int \theta_0 dx$ , then the non-positive solution to the Cauchy problem (1.1) can not be global in a time. This can be proven by borrowing the idea used in [4].

In fact, let  $\Theta = -\theta$  and  $\int_{\mathbb{R}^N} |x|^2 \Theta(x, t) dx = w(t)$ , then we have

$$\begin{aligned}
\frac{d}{dt} w(t) &= \int_{\mathbb{R}^N} |x|^2 \nabla \cdot (\mathcal{R}\Theta(x, t)\Theta(x, t)) dx \\
&= - \int_{\mathbb{R}^N} 2x(\mathcal{R}(\Theta)\Theta)(x, t) dx \\
&= -C_N \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(x-y) \cdot (x-y)\Theta(x, t)\Theta(y, t)}{|x-y|^{N-1}} dx dy. \\
&= -C_N \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\Theta(x, t)\Theta(y, t)}{|x-y|^{N-1}} dx dy.
\end{aligned} \tag{4.24}$$

where we have used the property (1.15) of the Riesz transform.

Let  $M = -\int_{\mathbb{R}^N} \theta_0 dx = \int_{\mathbb{R}^N} \Theta dx$ ,  $J_N = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x-y|^{-(N-1)} \Theta(x, t)\Theta(y, t) dx dy$ , then we have

$$\begin{aligned}
M^2 &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Theta(x, t)\Theta(y, t) dx dy \\
&= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (\Theta(x, t)\Theta(y, t))^{\frac{N-1}{N+1}} |x-y|^{\frac{2(N-1)}{N+1}} (\Theta(x, t)\Theta(y, t))^{\frac{2}{N+1}} |x-y|^{-\frac{2(N-1)}{N+1}} dx dy \\
&= \left( \int_{\mathbb{R}^2} \int_{\mathbb{R}^N} \Theta(x, t)\Theta(y, t) |x-y|^2 dx dy \right)^{\frac{N-1}{N+1}} \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Theta(x, t)\Theta(y, t) |x-y|^{-(N-1)} dx dy \right)^{\frac{2}{N+1}} \\
&= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Theta(x, t)\Theta(y, t) (|x|^2 - x \cdot y - y \cdot x + |y|^2) dx dy)^{\frac{N-1}{N+1}} J_N^{\frac{2}{N+1}} \\
&= (2Mw - 2 \left| \int_{\mathbb{R}^2} x \Theta(x, t) dx \right|^2)^{\frac{N-1}{N+1}} J_N^{\frac{2}{N+1}} \\
&\leq (2Mw)^{\frac{N-1}{N+1}} J_N^{\frac{2}{N+1}},
\end{aligned}$$

which implies

$$2^{-\frac{N-1}{2}} M^{\frac{N+3}{2}} w^{-\frac{N-1}{2}} \leq J_N, \tag{4.25}$$

Combing the above inequalities (4.24) and (4.25), we know

$$\frac{dw}{dt} \leq -C_N 2^{-\frac{N-1}{2}} M^{\frac{N+3}{2}} w^{-\frac{N-1}{2}}, \tag{4.26}$$

If we assume the right-hand side of the inequality (4.26) is strictly negative for  $t = 0$ , then it is always strictly negative for some finite  $t > 0$ . Hence  $w(t)$  will be negative for some finite  $t$ , which is a contradiction with  $w(t) \geq 0$ . This completes the proof of Remark 4.2.

## §5 Asymptotic behavior: The proof of Theorem 1.8

In this section, we prove Theorem 1.8 by using Fourier splitting method, which was used first by Schonbek [24, 25] and then used in [13, 33] to obtain decay rate in the context of the usual quasi-geostrophic equations. It should be pointed out that the present proofs could be extended to the system for the case  $\alpha \in (0, 2]$  provided there were on a priori bound of the derivatives of the solutions in the space  $L^2$ . For the global weak solution, the similar decay rate estimate can be also obtained by using the retarded mollification technique used in [5, 13, 24].

**Proof of the Theorem 1.8:** We will establish the decay estimate by employing the Fourier splitting method.

First we claim that  $\theta$  satisfies the following a priori estimate

$$|\hat{\theta}(\xi, t)| \leq \|\theta_0\|_{L^1} + |\xi| \int_0^t \|\theta(\tau)\|_{L^2}^2 d\tau. \quad (5.1)$$

In fact, we have from (1.1)

$$\partial_t \hat{\theta} + \nu |\xi|^{2\alpha} \hat{\theta} = -\widehat{\operatorname{div}(u\theta)}, \quad (5.2)$$

and we estimate the right-hand side of (5.2) as follows

$$|-\widehat{\operatorname{div}(u\theta)}| = |\widehat{\xi u \theta}| = |\xi| |\widehat{u \theta}| = |\xi| \|u\|_{L^2} \|\theta\|_{L^2} \leq |\xi| \|\theta(t)\|_{L^2}^2. \quad (5.3)$$

After integrating (5.2) and using (5.3), we obtain (5.1).

Now we want to obtain the decay estimate  $\|\theta(t)\|_{L^2}$ . Multiplying both sides of (1.1) by  $\theta(t)$  and integrating in  $\mathbb{R}^N$ , one get

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} |\theta|^2 dx + \nu \int_{\mathbb{R}^N} |\Lambda^{\frac{\alpha}{2}} \theta|^2 dx = - \int_{\mathbb{R}^N} \operatorname{div}(u\theta) \theta dx = - \frac{1}{2} \int_{\mathbb{R}^N} \theta^2 \Lambda \theta dx \leq 0, \quad (5.4)$$

which gives, by the Plancherel's theorem, that

$$\|\theta\|_{L^2} \leq \|\theta_0\|_{L^2}, \quad (5.5)$$

$$\frac{d}{dt} \int_{\mathbb{R}^N} |\hat{\theta}|^2 d\xi + 2\nu \int_{\mathbb{R}^N} |\xi|^\alpha |\hat{\theta}|^2 d\xi = - \int_{\mathbb{R}^N} \widehat{\operatorname{div}(u\theta)} \bar{\hat{\theta}} d\xi \leq 0. \quad (5.6)$$

Let introduce  $B(t) = \{\xi \in \mathbb{R}^N; |\xi| \leq M(t)\}$  with  $M(t) > 0$  to be determined appropriately below and  $B(t)^c$  is the complement of  $B(t)$ . By (5.5), we can estimate the second term in the left hand side of (5.6)

$$\int_{\mathbb{R}^N} |\xi|^\alpha |\hat{\theta}|^2 d\xi \geq \int_{B(t)^c} |\xi|^\alpha |\hat{\theta}|^2 d\xi \geq M^\alpha(t) \int_{B(t)^c} |\hat{\theta}|^2 d\xi = M^\alpha(t) \int_{\mathbb{R}^N} |\hat{\theta}|^2 d\xi - M^\alpha(t) \int_{B(t)} |\hat{\theta}|^2 d\xi, \quad (5.7)$$

Combining (5.6) and (5.7), using (5.1) and (5.5), we get

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^N} |\hat{\theta}|^2 d\xi + 2M^\alpha(t) \nu \int_{\mathbb{R}^N} |\hat{\theta}|^2 d\xi \\ & \leq CM^\alpha(t) \int_0^{M(t)} (\|\theta_0\|_{L^1} + r \int_0^t \|\theta(\tau)\|_{L^2}^2 d\tau)^2 r^{N-1} dr \\ & \leq CM^\alpha(t) \int_0^{M(t)} (\|\theta_0\|_{L^1}^2 + r^2 t \int_0^t \|\theta(\tau)\|_{L^2}^4 d\tau) r^{N-1} dr. \end{aligned} \quad (5.8)$$

Integrating (5.8), we get

$$\begin{aligned} & e^{2\nu \int_0^t M^\alpha(\tau) d\tau} \int_{\mathbb{R}^N} |\hat{\theta}|^2 d\xi \\ & \leq \|\theta_0\|_{L^2}^2 + C \int_0^t e^{2\nu \int_0^s M^\alpha(\tau) d\tau} (\|\theta_0\|_{L^1}^2 M^{N+\alpha}(s) \\ & \quad + s M^{N+2+\alpha}(s) \int_0^s \|\theta(\tau)\|_{L^2}^4 d\tau) ds. \end{aligned} \quad (5.9)$$

Now we take  $M^\alpha(t) = \frac{1}{2\beta\nu(t+1)}$  and thus  $e^{2\nu \int_0^t M^\alpha(\tau) d\tau} = (1+t)^{\frac{1}{\beta}}$ . From (5.9) and (5.5), we get

$$\begin{aligned}
& (1+t)^{\frac{1}{\beta}} \int_{\mathbb{R}^N} |\hat{\theta}|^2 d\xi \\
& \leq \|\theta_0\|_{L^2}^2 + C \int_0^t (1+s)^{\frac{1}{\beta}} \left\{ \|\theta_0\|_{L^1}^2 \left( \frac{1}{2\alpha\nu(1+s)} \right)^{\frac{N+\alpha}{\alpha}} \right. \\
& \quad \left. + s \left( \frac{1}{2\alpha\nu(s+1)} \right)^{\frac{N+2+\alpha}{\alpha}} \int_0^s \|\theta(\tau)\|_{L^2}^4 d\tau \right\} ds \\
& \leq \|\theta_0\|_{L^2}^2 + C \int_0^t \|\theta_0\|_{L^1}^2 \left( \frac{1}{2\beta\nu} \right)^{\frac{N+\alpha}{\alpha}} (1+s)^{-\frac{N+\alpha}{\alpha} + \frac{1}{\beta}} ds \\
& \quad + C \int_0^t \left( \frac{1}{2\beta\nu} \right)^{\frac{N+2+\alpha}{\alpha}} (1+s)^{-\frac{N+2+\alpha}{\alpha} + \frac{1}{\beta} + 2} ds \|\theta_0\|_{L^2}^4. \tag{5.10}
\end{aligned}$$

Since  $N > 2$  and  $1 \leq \alpha \leq 2$ , we take  $\frac{1}{\beta} = \frac{N+2-2\alpha}{\alpha} - \epsilon$  for some small  $\epsilon > 0$ , and hence  $\frac{1}{\beta} - \frac{N+2-2\alpha}{\alpha} < 0$  and  $\frac{1}{\beta} - \frac{N}{\alpha} < 0$ . Thus, from (5.10), we obtain

$$\begin{aligned}
& (1+t)^{-\frac{1}{\beta}} \|\theta(t)\|_{L^2}^2 \\
& \leq C \|\theta_0\|_{L^2}^2 + C \|\theta_0\|_{L^1}^2 \left( \frac{1}{2\beta\nu} \right)^{\frac{N+\alpha}{\alpha}} \frac{1}{\frac{N}{\alpha} - \frac{1}{\beta}} + C \left( \frac{1}{2\beta\nu} \right)^{\frac{N+2+\alpha}{\alpha}} \|\theta_0\|_{L^2}^4 \frac{1}{\frac{N+2-2\alpha}{\alpha} - \frac{1}{\beta}},
\end{aligned}$$

which gives the following decay rate in time

$$\|\theta(t)\|_{L^2}^2 \leq C(1+t)^{-\left(\frac{N+2-2\alpha}{\alpha} - \epsilon\right)} \tag{5.11}$$

for some  $\epsilon$  sufficiently small. Here the constant  $C$  depends upon  $L^1$  and  $L^2$  norms of  $\theta_0$ .

Next we obtain the decay estimate on  $\|\theta(t)\|_{L^p}$ ,  $p > 2$  by using the method used in [9].

Multiplying both sides of (1.1) by  $|\theta(t)|^{p-2}\theta(t)$ , integrating in  $\mathbb{R}^N$  and applying (1.12) in Lemma 1.12, one get

$$\frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}^N} |\theta|^p dx + \nu \frac{2}{p} \int_{\mathbb{R}^N} |\Lambda^{\frac{\alpha}{2}} |\theta|^{\frac{p}{2}}|^2 dx \leq - \int_{\mathbb{R}^N} \operatorname{div}(u\theta) |\theta|^{p-2} \theta dx = -\frac{1}{2} \int_{\mathbb{R}^N} |\theta|^p \Lambda \theta dx \leq 0. \tag{5.12}$$

Using Gagliardo-Nirenberg inequality, we have

$$\frac{d}{dt} \int_{\mathbb{R}^N} |\theta|^p dx \leq -2\nu C \left( \int_{\mathbb{R}^N} |\theta|^{\frac{pN}{N-\alpha}} dx \right)^{\frac{N-\alpha}{N}} \tag{5.13}$$

with  $C$  depending on  $\alpha$  and  $N$ . By interpolation we get

$$\|\theta\|_{L^p} \leq \|\theta\|_{L^2}^{1-\gamma} \left( \int_{\mathbb{R}^N} |\theta|^{\frac{pN}{N-\alpha}} dx \right)^{\gamma \frac{N-\alpha}{pN}}, \quad \gamma = \frac{N(p-2)}{N(p-2) + 2\alpha}. \tag{5.14}$$

Putting (5.14) into (5.13), we have

$$\frac{d}{dt} \|\theta(t)\|_{L^p}^p + 2C\nu \|\theta\|_{L^2}^{p-\frac{p}{\gamma}} \|\theta\|_{L^p}^{\frac{p}{\gamma}} \leq 0. \tag{5.15}$$

Since  $\gamma \in (0, 1)$  and  $\|\theta(t)\|_{L^2} \leq \|\theta_0\|_{L^2}$ , from (5.15), we obtain

$$\frac{d}{dt} \|\theta(t)\|_{L^p}^p + 2C\nu \|\theta_0\|_{L^2}^{p-\frac{p}{\gamma}} \|\theta\|_{L^p}^{\frac{p}{\gamma}} \leq 0. \tag{5.16}$$

which, by integration, give

$$\|\theta(t)\|_{L^p} \leq \|\theta_0\|_{L^p} \left( 1 + \frac{1-\gamma}{\gamma} \frac{2\nu C \|\theta_0\|_{L^2}^{\frac{p}{\gamma}-p}}{\|\theta_0\|_{L^2}^{\frac{p}{\gamma}-p}} t \right)^{-\frac{\gamma}{p(1-\gamma)}}. \tag{5.17}$$

Finally, we need to estimate  $\|\nabla \theta\|_{L^2}$ .

Multiplying both sides of (1.1) by  $\Lambda^2\theta(t)$  and integrating in  $\mathbb{R}^N$ , we get

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} \|\Lambda\theta\|_{L^2}^2 dx + \nu \int_{\mathbb{R}^N} \|\Lambda^{\frac{\alpha}{2}+1}\theta(t)\|^2 dx = - \int_{\mathbb{R}^N} \operatorname{div}(u\theta) \Lambda^2\theta dx. \quad (5.18)$$

The right-hand side of (5.18) can be estimated by

$$\begin{aligned} \left| \int_{\mathbb{R}^N} \operatorname{div}(u\theta) \Lambda^2\theta dx \right| &= \left| \int_{\mathbb{R}^N} (\xi_1 \widehat{u_1\theta}(\xi) + \xi_1 \widehat{u_2\theta}(\xi) + \cdots + \xi_1 \widehat{u_N\theta}(\xi)) |\xi|^2 \hat{\theta}(\xi) d\xi \right| \\ &\leq \sum_{i=1}^N \int_{\mathbb{R}^N} |\xi|^{2-\frac{\alpha}{2}} |\widehat{\theta u_i}(\xi)| |\xi|^{\frac{\alpha}{2}+1} |\hat{\theta}(\xi)| d\xi \\ &\leq \sum_{i=1}^N \|\Lambda^{2-\frac{\alpha}{2}}(\theta u_i)\|_2 \|\Lambda^{\frac{\alpha}{2}+1}\theta\|_2 \\ &\leq \frac{\nu}{4} \|\Lambda^{\frac{\alpha}{2}+1}\theta\|_2^2 + \frac{2}{\nu} \sum_{i=1}^N \|\Lambda^{2-\frac{\alpha}{2}}(\theta u_i)\|_2^2, \end{aligned} \quad (5.19)$$

where we have used the Plancherel and Hölder inequality.

By the fractional calculus inequality (1.10) with  $r = 2$  and Lemma 1.13, we have

$$\begin{aligned} \|\Lambda^{2-\frac{\alpha}{2}}(\theta u_i)\|_2 &\leq C(\|u_i\|_q \|\Lambda^{2-\frac{\alpha}{2}}\theta\|_p + \|\theta\|_q \|\Lambda^{2-\frac{\alpha}{2}}u_i\|_p) \\ &\leq C\|\theta\|_q \|\Lambda^{2-\frac{\alpha}{2}}\theta\|_p \end{aligned} \quad (5.20)$$

for  $i = 1, 2, \dots, N$  and  $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ . By the maximum principle  $\|\theta\|_{L^q} \leq C\|\theta_0\|_{L^q}$ , we have

$$\|\Lambda^{2-\frac{\alpha}{2}}(\theta u_i)\|_2 \leq C(\theta_0) \|\Lambda^{2-\frac{\alpha}{2}}\theta\|_p \quad (5.21)$$

for  $i = 1, 2, \dots, N$ .

Using Lemma 1.10, we get

$$\|\Lambda^{2-\frac{\alpha}{2}}\theta\|_p \leq C(\theta_0) \|\Lambda^{2-\frac{\alpha}{2}+\delta}\theta\|_2, \quad (i = 1, 2, \dots, N), \quad (5.22)$$

where  $\frac{1}{p} = \frac{1}{2} - \frac{\delta}{N}$  and  $0 < \delta < N$ .

Combining (5.18), (5.19), (5.20), (5.21) and (5.22), one get

$$\frac{1}{2} \frac{d}{dt} \|\Lambda\theta\|_{L^2}^2 + \frac{3}{4} \nu \|\Lambda^{\frac{\alpha}{2}}\theta\|_{L^2}^2 \leq C(\theta_0) \|\Lambda^{2-\frac{\alpha}{2}+\delta}\theta\|_{L^2}. \quad (5.23)$$

For the right-hand of (5.23), we have

$$\begin{aligned} &\|\Lambda^{2-\frac{\alpha}{2}+\delta}\theta(t)\|_{L^2}^2 \\ &= \int_{B(t)} |\xi|^{4-\alpha+2\delta} |\hat{\theta}(t)|^2 d\xi + \int_{B(t)^c} |\xi|^{4-\alpha+2\delta} |\hat{\theta}(t)|^2 d\xi \\ &\leq M^{4-\alpha+2\delta}(t) \|\theta(t)\|_{L^2}^2 + \int_{B(t)^c} \frac{|\xi|^{4-\alpha+2\delta}}{|\xi|^{2(\frac{\alpha}{2}+1)}} |\xi|^{\frac{\alpha}{2}+1} |\hat{\theta}(t)|^2 d\xi \\ &\leq M^{4-\alpha+2\delta}(t) \|\theta(t)\|_{L^2}^2 + \int_{B(t)^c} |\xi|^{-2(\frac{\alpha}{2}+1)+(4-\alpha+2\delta)} |\xi|^{\frac{\alpha}{2}+1} |\widehat{\theta}(t)|^2 d\xi \\ &\leq M^{4-\alpha+2\delta}(t) \|\theta(t)\|_{L^2}^2 + M^{2(1-\alpha+\delta)}(t) \|\Lambda^{\frac{\alpha}{2}+1}\theta(t)\|_{L^2}^2. \end{aligned} \quad (5.24)$$

Because  $\alpha > 1$ , we can choose  $M$  large enough such that  $M^{2(1-\alpha+\delta)}(t) < \frac{\nu}{4C(\theta_0)}$ . It follows from (5.24) that

$$C(\theta_0) \|\Lambda^{2-\frac{\alpha}{2}+\delta}\theta(t)\|_{L^2}^2 dx \leq C(\theta_0) M^{4-\alpha+2\delta} \|\theta(t)\|_{L^2}^2 + \frac{\nu}{4} \|\Lambda^{\frac{\alpha}{2}+1}\theta\|_{L^2}^2. \quad (5.25)$$

Putting (5.25) into (5.23), we have

$$\frac{d}{dt}\|\Lambda\theta\|_{L^2}^2 + \nu\|\Lambda^{1+\frac{\alpha}{2}}\theta\|_{L^2}^2 \leq C\|\theta(t)\|_{L^2}^2. \quad (5.26)$$

Moreover, we have

$$\begin{aligned} \|\Lambda^{\frac{\alpha}{2}+1}\theta\|_{L^2}^2 &\geq \int_{B(t)^c} |\xi|^{2(\frac{\alpha}{2}+1)} |\hat{\theta}|^2 d\xi \\ &\geq M^\alpha(t) \int_{B(t)^c} |\xi|^2 |\hat{\theta}|^2 d\xi \\ &= M^\alpha(t) \|\Lambda\theta\|_{L^2}^2 - M^\alpha(t) \int_{B(t)} |\xi|^2 |\hat{\theta}|^2 d\xi, \end{aligned}$$

which yields to

$$\|\Lambda^{\frac{\alpha}{2}+1}\theta\|_{L^2}^2 \geq M^\alpha(t) \|\Lambda\theta\|_{L^2}^2 - M^{\alpha+2}(t) \|\theta(t)\|_{L^2}^2. \quad (5.27)$$

Combining (5.26) and (5.27), we have

$$\frac{d}{dt}\|\Lambda\theta\|_{L^2}^2 + \nu M^\alpha(t) \|\Lambda\theta\|_{L^2}^2 \leq C\|\theta\|_{L^2}^2 + CM^{\alpha+2}(t) \|\theta\|_{L^2}^2, \quad \forall M. \quad (5.28)$$

It's obvious that we need to the obtained estimate  $\|\theta(t)\|_{L^2}^2$ .

Putting (5.11) into (5.28) and letting  $M(t) = M$  be a constant large enough such that  $M^{\alpha+2} > 1$ , we have

$$\frac{d}{dt}\|\Lambda\theta\|_{L^2}^2 + \nu M^\alpha \|\Lambda\theta\|_{L^2}^2 \leq CM^{\alpha+2}(1+t)^{-\frac{1}{\beta}}. \quad (5.29)$$

Then, by multiplying  $e^{M^\alpha t}$  on (5.29) and integrating with respect to  $t$ , we obtain

$$\begin{aligned} \|\Lambda\theta\|_{L^2}^2 &\leq e^{-M^\alpha t} \|\Lambda\theta_0\|_{L^2}^2 + CM^{\alpha+2} \int_0^t e^{-M^\alpha(t-s)} (1+s)^{-\frac{1}{\beta}} ds \\ &\leq e^{-M^\alpha t} \|\Lambda\theta_0\|_{L^2}^2 + CM^{\alpha+2} (1+t)^{-\frac{1}{\beta}}, \end{aligned} \quad (5.30)$$

where we have used the estimate

$$\int_0^t e^{-M^\alpha(t-s)} (1+s)^{-\frac{1}{\alpha}} ds \leq C(1+t)^{-\frac{1}{\alpha}}, \quad t > 0.$$

Thanks to the fact  $\|\nabla\theta\|_{L^2} = \|\Lambda\theta\|_{L^2}$ , it follows from (5.30) that

$$\|\nabla\theta\|_{L^2} \leq (1+t)^{-\frac{1}{2}(\frac{N+2-2\alpha}{\alpha}-\epsilon)}. \quad (5.31)$$

The estimates (5.11), (5.17) and (5.31) give the desire decay estimates. This completes the proof of Theorem 1.8.

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